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STRATEGIC LIQUIDITY PROVISION IN LIMIT ORDER MARKETS

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STRATEGIC LIQUIDITY PROVISION IN LIMIT ORDER MARKETS

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We characterize and prove the existence of Nash equilibrium in a limit order market with a finite number of risk-neutral liquidity providers. We show that if there is sufficient adverse selection, then pointwise optimization (maximizing in p for each q) in a certain nonlinear pricing game produces a Nash equilibrium in the limit order market. The need for a sufficient degree of adverse selection does not vanish as the number of liquidity providers increases. Our formulation of the nonlinear pricing game encompasses various specifications of informed and liquidity trading, including the case in which nature chooses whether the market-order trader is informed or a liquidity trader. We solve for an equilibrium analytically in various examples and also present examples in which the first-order condition for pointwise optimization does not define an equilibrium, because the amount of adverse selection is insufficient.

KEYWORDS: Market microstructure, limit orders, liquidity, market makers, dealers, trading game, nonlinear pricing.

1. INTRODUCTION

IN RECENT YEARS, the role of designated market makers has diminished, as security exchanges have moved to the electronic limit order book format. In an electronic limit order book, some market participants submit limit orders because they want to make specific transactions; others hope to profit from providing liquidity and hence play the role previously played by designated market makers. In this paper, we study the game played by risk-neutral liquidity providers who submit limit orders in anticipation of a marketable order arriving from a trader who wants to make a specific transaction.

According to Treynor (1971), "the essence of market making, viewed as a business, is that in order for the market maker to survive and prosper, his gains from liquidity-motivated transactors must exceed his losses to information-motivated transactions." This observation underlies most of the theory of market microstructure. Consistent with that theory, we assume that the trader desiring to make a specific transaction may either have information or may be liquidity motivated (or both). A special case of the model we study was studied earlier by Biais, Martimort, and Rochet (2000, 2012). In their model, the trader desiring to make a specific transaction has private information about the asset value, constant absolute risk aversion (CARA) utility, and an endowment of the asset that is also private information. In another example of our model, nature chooses whether the trader is an informed trader or a liquidity-motivated trader. This is a canonical example in market microstructure theory (e.g., Glosten and Milgrom (1985)).

We show that the first-order condition for pointwise optimization in a certain nonlinear pricing game defines an equilibrium under certain circumstances.

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The essence of our sufficient conditions is that the adverse selection faced by the liquidity providers should be sufficiently high. In the CARA example, the precise condition is that the elasticity of the expected gain from trade, as a function of the reciprocal of the hazard rate of the trader's type, should exceed 1. In a specific example of the "informed or liquidity trader" model, we show that our sufficient condition is equivalent to a sufficiently high probability of nature choosing the informed trader.

In a limit order market, the marginal price paid by a market buy order must be nondecreasing in the quantity of the order. This structure imposes possibly binding constraints on the liquidity providers. For example, a monopolist liquidity provider who is unconstrained in his pricing schedule will want to offer quantity discounts in certain circumstances (Maskin and Riley (1984), Biais, Martimort, and Rochet (2000)), which is impossible in a limit order market. Not surprisingly, the sufficient condition we present for pointwise optimization to define an equilibrium for an oligopoly is related to, though distinct from, the condition given in Biais, Martimort, and Rochet (2000) for a monopolist's optimum in a limit order market to be pointwise optimal. Both require the conditional expectation of the asset value to be sufficiently sensitive to the type of the trader submitting a marketable order.

The requirement that there should be sufficient adverse selection does not vanish as the number of liquidity providers increases. In a specific version of the CARA model in which there is insufficient adverse selection (Example 1C in Section 6), the solution of the first-order condition is not an equilibrium, regardless of the number of liquidity providers.[2] In an example of the informed or liquidity trader model (Example 2A in Section 6), the required probability of facing an informed trader actually increases as the number of liquidity providers increases.

We do not have a general answer to the question of whether an equilibrium of any sort exists when there is a low degree of adverse selection. However, it seems likely that some assumption about adverse selection is needed in general. Consider the following simple example (we thank Dan Bernhardt for this example). Suppose nature chooses whether the trader submitting a marketable order is informed or liquidity motivated. Suppose that if the liquidity motivated trader wants to buy, then he wants to buy $\tilde{q}$ shares regardless of the cost, where $\tilde{q}$ is a random variable with support $[0, K]$ for some $K < \infty$. Suppose also that the asset value, which is known to the informed trader, has a distribution with unbounded support. Then liquidity providers will not offer more than K shares in aggregate, because shares at a depth exceeding K can only be sold to the informed trader, ensuring losses. However, if the total number of shares offered is less than or equal to K, then anyone offering shares can

[2]This example is a counterexample to the existence result of Biais, Martimort, and Rochet (2000), acknowledged in Biais, Martimort, and Rochet (2012).

make higher expected profits by offering the same number of shares at an approximately infinite price, ensuring that, with probability near 1, they are only sold to the liquidity-motivated trader at an approximately infinite profit. Thus, there is no equilibrium. The problem is that there is insufficient adverse selection at very high prices. A similar issue arises in Glosten's (1994) analysis of a competitive limit order book. Glosten's Assumption 2 rules out the possibility that extreme marginal valuations might only come from uninformed traders (see the example in Glosten (1994, p. 1137)).

In the context of the CARA model, the first-order condition for pointwise optimization is equivalent to a differential equation derived by Biais, Martimort, and Rochet (2000). Our sufficient conditions are different from those of Biais, Martimort, and Rochet (2012), even for the CARA model. As an example of the difference, we show that the first-order condition (FOC) defines an equilibrium when the agent's type in the CARA model is normally distributed and there is sufficient adverse selection. The CARA model with normal distributions is an important example in market microstructure (e.g., Glosten (1989, 1994)). For several reasons, the sufficient conditions in Biais, Martimort, and Rochet (2012) do not include normal distributions. The most important of these reasons is that their approach is based on

$$\begin{aligned}\text{some conditions} &\Rightarrow \text{solution of FOC is convex}\\ &\Rightarrow \text{solution of FOC is an equilibrium.}\end{aligned}$$

In the normal case, the solution to the first-order condition is strictly concave, yet, it is nevertheless an equilibrium. In the CARA–Normal model, our sufficient condition in terms of adverse selection takes a particularly simple form. We show that the elasticity condition is satisfied if the variance of private information exceeds the variance of risk-adjusted inventory risk. This is equivalent to the beta in the projection of the asset value on the trader's type being larger than 1/2 (a similar condition is needed for the monopolist's optimum in this model to be pointwise optimal; the proof of Lemma 2 in Appendix A uses the condition $\beta > 0.345$). Section 4 discusses the relation of the sufficient conditions of Biais, Martimort, and Rochet (2012) to our elasticity condition.

2. MODEL

We consider a game among n strategic traders (liquidity providers). These traders simultaneously submit collections of limit orders as defined below. Then an order arrives to the market from an $(n+1)$th trader. This order executes against the limit orders and thereby determines the quantities traded and the cash transfers made. The cash transfers are based on discriminatory pricing: a marketable order executes against existing limit orders at their limit prices. After the marketable order is executed, the liquidation value $\tilde{v}$ of the asset is revealed. The profits of the traders are thereby determined. We assume the n traders are risk neutral, though the $(n+1)$th trader may be risk averse. We also assume $\mathsf{E}|\tilde{v}| < \infty$.

For the sake of simplicity, and without loss of generality, we focus on the offer side of the market. The liquidity providers are really playing two games simultaneously: one in which they make bids to buy shares and one in which they offer shares to sell. These games are not truly separable, because bids and offers could cross, causing execution before the $(n+1)$th trader arrives. Of course, crossing cannot occur with symmetric strategies. Absent crossing—that is, assuming all offer prices are higher than all bid prices—we assume the decision of the $(n+1)$th trader of whether to buy and how much to buy depends only on the prices at which shares are offered and does not depend on the profile of bids of the liquidity providers. Likewise, we assume the selling decision of the $(n+1)$th trader depends only on the prices at which he can sell. These assumptions are satisfied in each of the three examples we will analyze. Under these assumptions, a symmetric equilibrium of the pair of games must have the property that the profile of bids is a Nash equilibrium in the bidding game and the profile of offers is a Nash equilibrium in the offering game. We analyze the Nash equilibrium in the offering game.

A collection of limit orders can be modeled as a transfer schedule T_i, where, for $q > 0$, $T_i(q)$ is the cash transfer required by trader i in order to supply q shares. A transfer schedule is defined to be a lower semicontinuous convex function $T_i : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ with $T_i(0) = 0$. If $T_i(q) = \infty$, then the trader is refusing to supply q shares at any price. As discussed by Biais, Martimort, and Rochet (2000), convexity guarantees that the transfer schedule can be interpreted as a limit order book. Specifically, convexity implies that there is a unique right-continuous nondecreasing function $P_i : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ such that $T_i(q) = \int_0^q P_i(x)\,dx$ for all q (Rockafellar (1970)). This function is the right-hand derivative of T_i, with $P_i(q) = \infty$ if $T_i(q) = \infty$. The price $P_i(q)$ is the limit price on the marginal share offered by trader i at a depth of q shares. We call P_i the price function associated with T_i.

Instead of studying the transfer schedule, we are going to work primarily with the associated offer curve. An offer curve is a right-continuous nondecreasing function $S : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$. The quantity $S(p)$ is the number of shares offered at prices less than or equal to p. The offer curve S_i associated with T_i is the right-continuous inverse of the price function P_i associated with T_i. It is defined by $S_i(p) = \inf\{q \mid P_i(q) > p\}$. The offer curve S_i can have discontinuities. If $S_i(p) - \lim_{y\uparrow p} S_i(y) = \Delta > 0$, then there is a discrete offer of size Δ at p. The offer is not all or nothing: trader i is offering Δ shares or fewer at price p; thus, supply is really a correspondence.

Given an offer curve S_i, we can define the transfer schedule with which S_i is associated as $T_i(q) = \int_0^q P_i(x)\,dx$, where P_i is defined as $P_i(x) = \inf\{p \mid S_i(p) > x\}$. So, S_i, P_i, and T_i are equivalent representations of a strategy. Also, a transfer schedule is strictly convex if and only if the associated price function is strictly increasing, which is equivalent to the associated offer curve being continuous. As in Biais, Martimort, and Rochet (2000), let T_{-i} denote the infimal convolution of the T_j for $j \neq i$, meaning that

$$T_{-i}(q) = \min\left\{ \sum_{j\neq i} T_j(q_j) \,\middle|\, \sum_{j\neq i} q_j = q, (\forall j)\, q_j \geq 0 \right\}.$$

The function T_{-i} is a transfer schedule, and the minimum in its definition is attained for each q. Let P_{-i} denote the price function and let S_{-i} denote the offer curve associated to T_{-i}.

Let Q_i denote the random quantity transacted by trader i for $i = 1, \dots, n$. The transactions are with the $(n+1)$th trader. We do not regard the $(n+1)$th trader as a strategic trader (i.e., as a participant in the game), though it would be possible to do so, at least in our first example (the CARA case). How $(Q_1, \dots, Q_n)$ depends on the profile of offer curves is part of the definition of the game played by the n strategic traders. This is specified below in three different examples.

Trader i wants to maximize

$$\text{(1)}\qquad \mathsf{E}\big[T_i(Q_i) - \tilde{v}Q_i\big] = \mathsf{E}\int_0^\infty \big[P_i(q_i) - \tilde{v}\big] 1_{\{Q_i > q_i\}}\, dq_i$$
$$= \int_0^\infty \mathsf{E}\big[\big[P_i(q_i) - \tilde{v}\big] 1_{\{Q_i > q_i\}}\big]\, dq_i,$$

where 1_A denotes the zero–one indicator function of an event A. A Nash equilibrium is a profile $(T_1, \dots, T_n)$ of transfer schedules such that T_i maximizes (1) for each i, taking T_j as given for $j \neq i$. Given the one-to-one correspondences between transfer schedules, price functions, and offer curves, we can equivalently define Nash equilibrium in terms of profiles of price functions or profiles of offer curves.

ASSUMPTION 1: *There exists a function* $u: \mathbb{R} \times [0, \infty] \to \mathbb{R}$ *that is absolutely continuous on* $\mathbb{R} \times [0, \infty)$ *and has the property that for all* i *and all profiles* $(T_1, \dots, T_n)$ *such that* S_{-i} *is continuous*,

$$\mathsf{E}\big[\big[P_i(q_i) - \tilde{v}\big] 1_{\{Q_i > q_i\}}\big] = u\big(P_i(q_i), q_i + S_{-i}\big(P_i(q_i)\big)\big).$$

Absolute continuity means that there exist functions g and h such that

$$u(p, q) = u(0, q) + \int_0^p g(x, q)\, dx = u(p, 0) + \int_0^q h(p, y)\, dy$$

for each (p, q). We will write u_p for g and u_q for h. The partial derivatives of u exist almost everywhere, and when they do, they equal u_p and u_q. We require u to be defined for $q = \infty$, because ∞ may be in the range of S_{-i}.

The condition that S_{-i} is continuous means that there are no flats (discrete orders) in the offer curves of other liquidity providers. When optimizing (1), we will allow trader i to post discrete orders if he so chooses. However, by relying on Assumption 1, we are implicitly limiting our search for equilibria to those in which there are no discrete orders. Thus, we are limiting our search to equilibria in which the nonincreasing marginal price constraint is not binding. To generalize the model, we would need to consider tie-breaking rules when multiple traders post discrete limit orders and some or all of those orders are only partially filled.

Assumption 1 holds in the CARA model (Example 1), but it is also more general, as Examples 2 and 3 illustrate. Example 2 captures the informed trader/liquidity trader dichotomy in the earlier quote from Treynor (1971). That dichotomy is standard in the market microstructure literature (e.g., Kyle (1985), Glosten and Milgrom (1985), Back and Baruch (2007)). Example 3 appears in Rock (1990) and Sandås (2001). We study specific versions of these examples in Section 6.

EXAMPLE 1—CARA Investor: Suppose the marketable order comes from an investor with constant absolute risk aversion α who has a random endowment $\tilde{w}$ of the asset and a private signal $\tilde{z}$. Assume $\tilde{v} = \tilde{z} + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is normally distributed, independent of $\tilde{z}$ and $\tilde{w}$, and has mean zero. Set $\gamma = \alpha \operatorname{var}(\tilde{\varepsilon})$. This investor chooses q to maximize $J(q, \tilde{w}, \tilde{z}) - T(q)$, where J is the certainty equivalent defined as

$$J(q, w, z) = (q + w)z - \frac{1}{2}\gamma(q + w)^2.$$

Set $\tilde{\theta} = \tilde{z} - \gamma\tilde{w}$. Letting Q denote an optimal trade size for the investor and letting $(Q_1, \ldots, Q_n)$ denote an optimal allocation of the trade among the limit order traders, we have, for any q_i,

$$\begin{aligned} Q_i > q_i \quad &\Leftrightarrow \quad J_q\big(q_i + S_{-i}(P_i(q_i)), \tilde{w}, \tilde{z}\big) > P_i(q_i) \\ &\Leftrightarrow \quad \tilde{\theta} > P_i(q_i) + \gamma\big[q_i + S_{-i}\big(P_i(q_i)\big)\big]. \end{aligned}$$

Assume the support of $\tilde{\theta}$ is an interval, possibly extending to $-\infty$ and/or $+\infty$, and assume $\tilde{\theta}$ has a density function f that is positive on the interior of the interval. Define a function v by $v(\theta) = \mathsf{E}[\tilde{v} \mid \tilde{\theta} = \theta]$. Define $u(p, \infty) = 0$ and

$$u(p, q) = \int_{p+\gamma q}^{\infty} \big(p - v(\theta)\big) f(\theta)\, d\theta \tag{2}$$

for $q < \infty$. Assumption 1 holds in this example for this function u.

EXAMPLE 2—Informed or Noise: Suppose the $(n+1)$th trader is randomly selected as either an informed trader or a liquidity (noise) trader. Assume the informed trader has observed the realization of $\tilde{v}$ and buys all quantities offered at prices less than $\tilde{v}$. We allow either a continuous or a discrete distribution for $\tilde{v}$. Let F denote the distribution function of $\tilde{v}$ and let $\bar{v}$ denote the mean of $\tilde{v}$. Conditional on the informed trader being chosen,

$$\mathsf{E}\big[[P_i(q_i)-\tilde{v}]1_{\{Q_i>q_i\}}\big]=\mathsf{E}\big[\big(P_i(q_i)-\tilde{v}\big)1_{\{\tilde{v}>P_i(q_i)\}}\big] = \int_{P_i(q_i)}^{\infty}\big(P_i(q_i)-v\big)\,dF(v). \tag{3}$$

Assume the liquidity trader submits a limit order for $\tilde{q}$ shares at limit price $\tilde{p}$. A negative $\tilde{q}$ represents a sell order. To include the possibility of a market buy order, we can allow $\tilde{p}=\infty$ with positive probability. We assume that $\tilde{p}$ and $\tilde{q}$ are independent of $\tilde{v}$ and independent of each other conditional on the sign of $\tilde{q}$. Let G denote the distribution function of $\tilde{p}$ conditional on $\tilde{q}>0$ and let H denote the distribution function of $\tilde{q}$ conditional on $\tilde{q}>0$. Conditional on the liquidity trader being chosen,

$$\mathsf{E}\big[[P_i(q_i)-\tilde{v}]1_{\{Q_i>q_i\}}\big]=\big(P_i(q_i)-\bar{v}\big)\mathsf{E}[1_{\{Q_i>q_i\}}] = \big(P_i(q_i)-\bar{v}\big)\mathsf{E}[1_{\{\tilde{p}>P_i(q_i),\tilde{q}>q_i+S_{-i}(P_i(q_i))\}}].$$

Due to the assumed independence, this equals

$$\big(P_i(q_i)-\bar{v}\big)\big(1-G\big(P_i(q_i)\big)\big)\big(1-H\big(q_i+S_{-i}\big(P_i(q_i)\big)\big)\big). \tag{4}$$

Let ϕ denote the probability the informed trader is chosen to trade, and assume the uninformed liquidity trader is chosen and submits a buy order ($\tilde{q}>0$) with probability $(1-\phi)/2$. Combining (3) and (4) shows that Assumption 1 holds with

$$u(p,q)=\phi\int_p^{\infty}(p-v)\,dF(v) + \frac{1-\phi}{2}(p-\bar{v})\big(1-G(p)\big)\big(1-H(q)\big),$$

taking

$$u(p,\infty)=\phi\int_p^{\infty}(p-v)\,dF(v).$$

EXAMPLE 3—Inelastic Demand: Assume a market order of a size $\tilde{q}$ that does not depend on the transfer schedules. Let F denote the distribution function of $\tilde{q}$ and assume it has a density f. Set $v(q) = \mathsf{E}[\tilde{v} \mid \tilde{q} = q]$. Assumption 1 holds for $u(p, \infty) = 0$ and

$$u(p, q) = \int_q^\infty \big(p - v(x)\big) f(x)\, dx$$

for $q < \infty$.

3. THE FIRST-ORDER CONDITION

It follows from Assumption 1 that, when S_{-i} is continuous, trader i chooses P_i to maximize

$$\int_0^\infty u\big(P_i(q_i), q_i + S_{-i}\big(P_i(q_i)\big)\big)\, dq_i. \tag{5}$$

Suppose that $S_j = S^*$ for all $j \neq i$ and some offer curve S^*. Suppose S^* is differentiable with derivative s^*. Then the first-order condition for pointwise maximization of (5)—maximizing in p for each q_i—is

$$u_p\big(p, q + (n-1)S^*(p)\big) + (n-1)s^*(p)u_q\big(p, q + (n-1)S^*(p)\big) = 0. \tag{6}$$

Imposing the first-order condition at $q = S^*(p)$ yields

$$u_p\big(p, nS^*(p)\big) + (n-1)s^*(p)u_q\big(p, nS^*(p)\big) = 0. \tag{7}$$

This is a differential equation that should hold for any symmetric equilibrium in which the optimum for each liquidity provider is pointwise optimal. In the setting of Example 1, this differential equation is equivalent to the differential equation (43) of Biais, Martimort, and Rochet (2000). The equivalence is demonstrated in Appendix B (thus, the differential equation (43) of Biais, Martimort, and Rochet (2000) characterizes equilibrium only when the liquidity providers' strategies are pointwise optimal).

Denoting the aggregate offer curve by $Q(p) = nS^*(p)$ and setting $\lambda = (n-1)/n$, the differential equation (7) is equivalent to

$$u_p\big(p, Q(p)\big) + \lambda u_q\big(p, Q(p)\big)\frac{dQ(p)}{dp} = 0. \tag{8}$$

There are monopoly and competitive versions of this equation. Pointwise maximization by a monopolist produces the first-order condition $u_p(p, Q(p)) = 0$. This is the special case $n = 1 \Leftrightarrow \lambda = 0$ of (8). Perfect competition means zero expected profits, that is, $u(p, Q(p)) = 0$. This equation implies that prices are tail expectations, as discussed in Glosten (1994). Differentiating it with respect to p produces equation (8) with $\lambda = 1$.

4. SECOND-ORDER CONDITIONS

In the objective function (5), adjusting the limit price p at any depth q_i has two effects: there is a direct effect measured by the partial derivative u_p and an indirect effect via the loss of priority in the limit order book. The cost of losing priority is the term $(n-1)s^*(p)u_q(p, q_i + (n-1)S^*(p))$ in the first-order condition (6). Along an equilibrium aggregate offer curve $q = nS^*(p)$, we expect that $u_p(p, q) > 0$ and $u_q(p, q) < 0$. The condition $u_p(p, q) > 0$ means that a monopolist would want to charge a higher price. The condition $u_q(p, q) < 0$ means that an increase in the quantity supplied by other limit order traders at prices at or below p (a reduction in priority for trader i) reduces the expected profit of trader i. With $u_p > 0$ and $u_q < 0$, the differential equation (7) implies that the offer curve has a positive slope (recall that it cannot be negative, by the nature of a limit order book). The sign of

$$u_p(p, q) + (n-1)s^*(p)u_q(p, q) \tag{9}$$

to the left and right of the solution of the differential equation is critical for determining whether the solution is an equilibrium.

THEOREM 1: *Assume S^* and p^{ask} satisfy the following statement: S^* is a continuous nondecreasing function that is positive and continuously differentiable for all $p > p^{\text{ask}}$ and is equal to zero for all $p \le p^{\text{ask}}$. Assume*

$$u_p(p, q) + (n-1)s^*(p)u_q(p, q) \begin{cases} \ge 0, & \text{if } q \ge nS^*(p), \\ \le 0, & \text{if } q \le nS^*(p), \end{cases} \tag{10}$$

for all p and all $q \ge 0$, where

$$s^*(p) = \begin{cases} \dfrac{dS^*(p)}{dp}, & \text{if } p > p^{\text{ask}}, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

Then the strategy profile $(S^*, \dots, S^*)$ is a Nash equilibrium.

Condition (10) implies that S^* satisfies the differential equation (7). The hypothesis (10) involves S^*. Thus, it does not allow us to verify the existence of equilibrium without first solving the differential equation. The following corollary also makes assumptions regarding the solution of the differential equation, but it provides a practical recipe for verifying condition (10). Because the solution of the differential equation lies between the competitive and monopoly solutions, condition (a) holds if $u_p(p, q) \ge 0$ for all quantities above the monopoly quantity and condition (b) holds if $u_q(p, q) \le 0$ for all quantities less than the competitive quantity. As the examples will illustrate, these are mild assumptions. Assumptions (c) and (d) are motivated by examples with bounded distributions. The hypotheses of those conditions typically hold outside the support of the distribution, where $u \equiv 0$; hence $u_p = 0$. They are also unrestrictive assumptions. Condition (e) holds if the ratio $u_p(p,q)/u_q(p,q)$ is a decreasing function of q. It is the most restrictive assumption and is discussed further below.

COROLLARY 1: *Assume S^* and p^{ask} satisfy the following statement: S^* is a continuous nondecreasing function that is positive and continuously differentiable for all $p > p^{\text{ask}}$ and is equal to zero for all $p \le p^{\text{ask}}$. Assume S^* satisfies the differential equation (7) for all $p > p^{\text{ask}}$. Define s^* by (11). Assume $u_p(p, nS^*(p)) > 0$ and $u_q(p, nS^*(p)) < 0$ for all p such that $s^*(p) > 0$. Also make the following assumptions.*

(a) *If $q > nS^*(p)$, then $u_p(p,q) \ge 0$.*
(b) *If $q < nS^*(p)$, then $u_q(p,q) \le 0$.*
(c) *If $q < nS^*(p)$ and $u_q(p,q) = 0$, then $u_p(p,q) \le 0$.*
(d) *If $q < nS^*(p)$ and $s^*(p) = 0$, then $u_p(p,q) \le 0$.*
(e) *For all p such that $s^*(p) > 0$,*

$$\frac{u_p(p,q)}{u_q(p,q)} \begin{cases} \le \dfrac{u_p(p,nS^*(p))}{u_q(p,nS^*(p))}, & \text{if } q > nS^*(p) \text{ and } u_q(p,q) < 0, \\[2ex] \ge \dfrac{u_p(p,nS^*(p))}{u_q(p,nS^*(p))}, & \text{if } q < nS^*(p) \text{ and } u_q(p,q) < 0. \end{cases} \tag{12}$$

Then the strategy profile $(S^*, \dots, S^*)$ is a Nash equilibrium.

We can also derive a necessary condition for equilibrium in terms of the sign of (9), which is expressed in the following theorem.

THEOREM 2: *Assume S^* and p^{ask} satisfy the following statement: S^* is a continuous nondecreasing function that is positive and continuously differentiable for all $p > p^{\text{ask}}$ and is equal to zero for all $p \le p^{\text{ask}}$. Assume S^* satisfies the differential equation (7) for all $p > p^{\text{ask}}$. Define s^* by (11). Assume there exists $\hat{p} > p^{\text{ask}}$ such that s^* is positive at $\hat{p}$ and a neighborhood K of $(\hat{p}, nS^*(\hat{p}))$ such that*

$$\begin{aligned} &(p,q) \in K \quad \text{and} \quad q > nS^*(p) \\ &\quad \Rightarrow \quad u_p(p,q) + (n-1)s^*(p)u_q(p,q) < 0 \end{aligned} \tag{13a}$$

or

$$\begin{aligned} &(p,q) \in K \quad \text{and} \quad q < nS^*(p) \\ &\quad \Rightarrow \quad u_p(p,q) + (n-1)s^*(p)u_q(p,q) > 0. \end{aligned} \tag{13b}$$

Then the strategy profile $(S^*, \dots, S^*)$ is not a Nash equilibrium.

Condition (13) typically occurs when the sign is reversed in assumption (e) of Corollary 1. We state this as a corollary to Theorem 2. Example 1C in the next section applies the corollary.

COROLLARY 2: *Assume* S^* *and* p^{ask} *satisfy the following statement*: S^* *is a continuous nondecreasing function that is positive and continuously differentiable for all* $p > p^{\text{ask}}$ *and is equal to zero for all* $p \le p^{\text{ask}}$. *Assume* S^* *satisfies the differential equation* (7) *for all* $p > p^{\text{ask}}$. *Define* s^* *by* (11). *Assume there exists* $\hat{p} > p^{\text{ask}}$ *such that* $s^*(\hat{p}) > 0$, u *is continuously differentiable on a neighborhood of* $(\hat{p}, nS^*(\hat{p}))$, $u_q(\hat{p}, nS^*(\hat{p})) < 0$, *and*

$$\frac{d}{dq}\frac{u_p(p,q)}{u_q(p,q)} > 0 \tag{14}$$

evaluated at $p = \hat{p}$ *and* $q = nS^*(\hat{p})$. *Then the strategy profile* $(S^*, \dots, S^*)$ *is not a Nash equilibrium*.

To illustrate conditions (12) and (14), consider Example 1. Set $x = p + \gamma q$. We have

$$u_p(p,q) = \big(v(x) - p\big)f(x) + 1 - F(x),$$
$$u_q(p,q) = \gamma\big(v(x) - p\big)f(x).$$

Set $R(x) = (1 - F(x))/f(x)$ when $f(x) > 0$. For x in the support of $\tilde{\theta}$, we have

$$\frac{d}{dq}\left(\frac{u_p(p,q)}{u_q(p,q)}\right) < 0 \quad \Leftrightarrow \quad \big(p - v(x)\big)R'(x) + R(x)v'(x) > 0. \tag{15}$$

Biais, Martimort, and Rochet (2012) assume $R' < 0$. It is actually easier to satisfy the second-order condition (10) when $R' \ge 0$. This includes exponential distributions, for which R is constant, as well as hyperexponential distributions and others. When $R' \ge 0$, condition (15) holds on the support of $\tilde{\theta}$, provided only that v is monotone and $p > v(x)$, which is certainly true whenever p is above the competitive price (the upper-tail expectation). The same is true in Example 3, replacing $\tilde{\theta}$ with $\tilde{q}$. See Examples 1A and 3A for more details.

Now consider the case that $R' < 0$. Then, fixing p, R is decreasing in q and $p - v$ is also decreasing in q, so we can invert the function $q \mapsto R(p + \gamma q)$ and write the gain from trade $p - v$ as a increasing function of R. Define the elasticity

$$\frac{d\log(p - v)}{d\log R} \equiv \frac{\partial \log(p - v(p + \gamma q))/\partial q}{\partial \log R(p + \gamma q)/\partial q}. \tag{16}$$

When $p > v(x)$ and $R' < 0$, then the equivalent conditions in (15) are equivalent to this elasticity being larger than 1. This is true when the expected asset value $v(\theta)$ is relatively sensitive to the trader's type θ, in other words, when there is a relatively high degree of adverse selection.

Biais, Martimort, and Rochet (2012) give alternative sufficient conditions for the differential equation (7) to define an equilibrium, in the context of the CARA model. Their results are all based on their lemma that shows that the solution of the differential equation (7) is an equilibrium if S^* is convex. Suppose S^* satisfies (7), and set

$$\omega(p, q) = u_p(p, q) + (n-1)s^*(p)u_q(p, q). \tag{17}$$

Under the assumptions of Biais, Martimort, and Rochet (2012), which include $v' < 1$ and $R' < 0$, convexity of S^* implies that $\omega(p, q)/f(p + \gamma q)$ is a decreasing function of p at each (p, q) such that $s^*(p) > 0$ and $p > v(p + \gamma q)$. So, in the usual qp plane, $\omega(p, q)$ is negative above the graph of nS^* and positive below, at least down to the price $p = v(p + \gamma q)$, which is below the competitive price (the upper-tail expectation) and beyond which deviations cannot be profitable. Being negative above and positive below is the same as being negative to the left and positive to the right, which is the assumption in our Theorem 1. However, Theorem 1 is more general than convexity of S^*. We show in the next section that it applies when $\tilde{\theta}$ is normally distributed and there is sufficient adverse selection, a setting in which S^* is concave.

To deduce that S^* is convex, Biais, Martimort, and Rochet (2012) impose assumptions on exogenous variables. The most straightforward of those results is their Corollary 1, which assumes that v and R are concave. Concavity of v and R combined with the hypotheses $v' > 0$ and $R' < 0$ implies that

$$\big(p - v(\theta)\big)R'(\theta) + R(\theta)v'(\theta) \tag{18}$$

is a decreasing function of θ at each (p, θ) such that $p > v(\theta)$. Biais, Martimort, and Rochet (2012) assume there is a maximum type $\bar{\theta}$. The expression (18) is positive at $\bar{\theta}$ and hence positive at $\theta \le \bar{\theta}$ if $v(\theta) < p < v(\bar{\theta})$. This means that the elasticity (16) is greater than 1 at all (p, q) such that $v(p + \gamma q) < p < v(\bar{\theta})$. The boundary condition in Biais, Martimort, and Rochet (2012) implies that the marginal price paid by the trader of type $\bar{\theta}$ is $v(\bar{\theta})$. Therefore, under these hypotheses, the elasticity (16) is greater than 1 at all prices at which transactions take place. However, the next section shows that the elasticity can be greater than 1 even when v is linear and R is convex, which is inconsistent with the hypotheses of both Corollary 1 and Proposition 1 of Biais, Martimort, and Rochet (2012).

5. CARA–NORMAL EXAMPLE

This section analyzes Example 1 when the $(n+1)$th trader's type is normally distributed. Recall that we have $\tilde{\theta} = \tilde{z} - \gamma\tilde{w}$ and $v(\theta) = \mathsf{E}[\tilde{z} \mid \tilde{\theta} = \theta]$. Assume

$\tilde{z}$ and $\tilde{w}$ are independent and normally distributed, and $\tilde{w}$ has zero mean.[3] In this circumstance, $\tilde{\theta}$ is normally distributed, and $\tilde{\theta}$, $\tilde{v}$, and $\tilde{z}$ all have the same means. Denote the common mean by $\bar{v}$. We have $v(\theta) = \bar{v} + \beta(\theta - \bar{v})$, with

$$\beta = \frac{\mathrm{var}(\tilde{z})}{\mathrm{var}(\tilde{z}) + \mathrm{var}(\gamma\tilde{w})}.$$

Assume $\mathrm{var}(\tilde{z}) > \mathrm{var}(\gamma\tilde{w})$, so $1/2 < \beta < 1$. This assumption implies a sufficient degree of adverse selection, enabling us to establish that the elasticity (16) is larger than 1. Denote the variance of $\tilde{\theta}$ by σ^2.

Recall that F and f denote the distribution and density functions of $\tilde{\theta}$, and $R(x) = (1 - F(x))/f(x)$. Set $x = p + \gamma q$. We have

$$\begin{aligned} u(p, q) &= \int_x^\infty \big(p - \bar{v} - \beta(y - \bar{v})\big) f(y)\, dy \\ &= (p - \bar{v})\big(1 - F(x)\big) - \beta\sigma^2 f(x), \end{aligned} \tag{19}$$

using $f'(y) = -(y - \bar{v})f(y)/\sigma^2$ for the second equality. Note that

$$u(p, q) > 0 \quad \Leftrightarrow \quad p > \bar{v} + \frac{\beta\sigma^2}{R(x)}.$$

The right-hand side of the right-hand inequality is $\mathsf{E}[\tilde{v} \mid \tilde{\theta} \geq x]$. The following lemma is certainly known, but we provide a proof in the Appendix for completeness.

LEMMA 1: *R is a decreasing convex function that satisfies*

$$-\frac{R(x)}{R'(x)} > x - \bar{v} \tag{20}$$

for all x. Moreover, $R(\bar{v} + \sigma) > \sigma/2$ and $\lim_{x\to\infty}(x - \bar{v})R(x) = \sigma^2$.

The inequality (20) is used below in conjunction with the assumption $\beta > 1/2$ to establish that the elasticity (16) is larger than 1.

LEMMA 2: *For each $0 \leq \lambda \leq 1$, there exists p_λ and a continuous nondecreasing function Q_λ such that Q_λ is positive and satisfies the differential equation (8) for all $p > p_\lambda$ and satisfies $Q_\lambda(p) = 0$ for all $p \leq p_\lambda$. Furthermore, $Q_a(p) \geq Q_b(p)$ for all p when $a > b$. Also, $u_p(p, Q_0(p)) = 0$ for all $p > p_0$ and $u(p, Q_1(p)) = 0$ for all $p > p_1$.*

[3]The $(n+1)$th trader shifts inventory risk to the liquidity providers due to their different risk aversions. The zero-mean assumption implies that liquidity providers do not expect to end up with either positive or negative inventory. The same assumption was made by Glosten (1989, 1994).

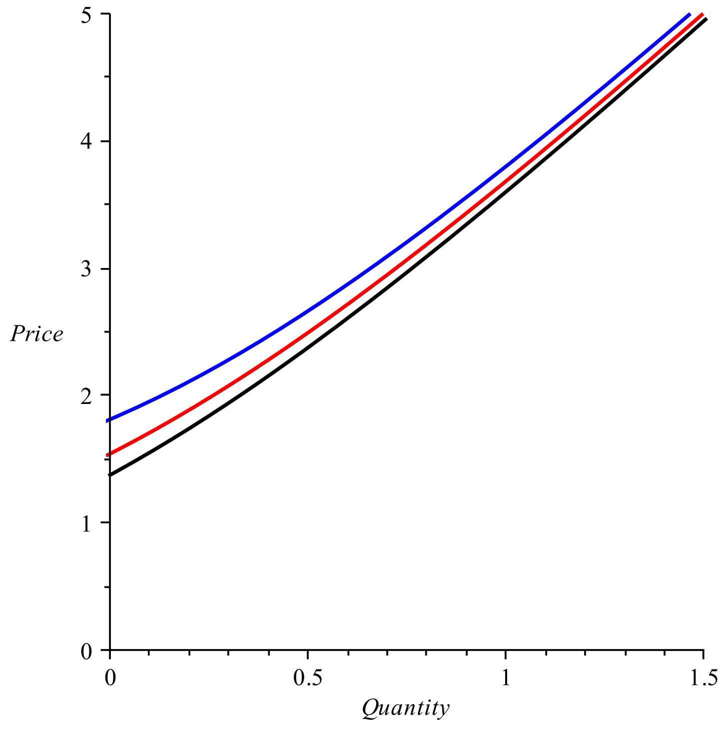

FIGURE 1.—Aggregate offer curves in the CARA–Normal example. These are the aggregate offer curves in Example 1 when $\tilde{\theta}$ has a standard normal distribution and $v(\theta) = 3\theta/4$. The bottom curve is the perfectly competitive solution, the middle curve is for $n = 2$, and the top curve is the monopoly solution.

THEOREM 3: *Set $S^*(p) = Q_\lambda(p)/n$ with $\lambda = (n-1)/n$, where Q_λ is defined in Lemma 2. The profile $(S^*, \dots, S^*)$ of offer curves is a Nash equilibrium.*

The equilibrium aggregate offer curve is depicted in Figure 1. Note that quantity is a concave function of price; equivalently, price is a convex function of quantity. This means that the third derivative of the transfer schedule is positive.

6. ADDITIONAL EXAMPLES

In each example, we want to find a solution of the differential equation (7) that satisfies the second-order conditions in Theorem 1 or Corollary 1. We do this by solving the differential equation subject to the boundary conditions (21) stated below. The practical motivation for these boundary conditions is that they work, as the examples illustrate. An intuitive motivation for them is as follows. Suppose there is a maximum quantity $q^{\max}$ that is transacted in equilibrium. A game in which each liquidity provider chooses the optimal price at which to sell $q^{\max}/n$ is a Bertrand game. The price should be optimal under the assumption of no supply response from other traders, meaning

$$u_p(p^{\max}, q^{\max}) = 0. \tag{21a}$$

Also, because it is Bertrand, there should be zero expected profits at the quantity $q^{\max}$. This means

$$u(p^{\max}, q^{\max}) = 0. \tag{21b}$$

In examples with bounded distributions, we find $p^{\max}$ and $q^{\max}$ by solving the boundary conditions (21). We then solve the differential equation (7) backward until we reach $q = 0$. In the setting of Example 1, the boundary conditions (21) are equivalent to the boundary condition of Biais, Martimort, and Rochet (2000); see Appendix B.

Notice that the boundary condition (21) is independent of n. So when distributions are bounded, we are looking for solutions of (7) that intersect at a common point $(p^{\max}, q^{\max})$. The dependence on the parameter n implies that the solutions are necessarily ordered, with supply being larger for larger n.

EXAMPLE 1A: In Example 1, assume that $\tilde{\theta}$ is exponentially distributed with parameter τ and that v is strictly monotone. Then the equivalent conditions in (15) hold. This implies the second-order condition (10) in Theorem 1 at each p such that $s^*(p) > 0$. Thus, a solution of the differential equation (7) that is strictly increasing above p^{ask} is an equilibrium, provided only that $u_p(p, q) \geq 0$ for $p < p^{\text{ask}}$. For a specific example, take $v(q) = \beta q$ for $0 < \beta < 1$. There is a unique affine solution $Q_\lambda(p) = A + Bp$ of the differential equation (8) with coefficients

$$A = -\frac{1}{\tau[(1-\lambda)\beta\gamma + \lambda\gamma]}, \quad B = \frac{1-\beta}{\beta\gamma}.$$

The solution p^{ask} of $Q_\lambda(p^{\text{ask}}) = 0$ satisfies

$$p^{\text{ask}} < \frac{1}{(1-\beta)\tau}.$$

Given this fact, we can calculate that

$$p < p^{\text{ask}} \quad \Rightarrow \quad u_p(p, q) \geq 0.$$

Thus, setting $S^* = Q_\lambda/n$ for $\lambda = (n-1)/n$, the profile $(S^*, \ldots, S^*)$ is a Nash equilibrium.

EXAMPLE 1B: In Example 1, assume $\tilde{\theta}$ is uniformly distributed on $(0, 1)$ and $v(x) = \beta x$ for all x, where $0 < \beta < 1$. This example is also analyzed in Biais, Martimort, and Rochet (2012). We have

$$u(p, q) = p(1 - p - \gamma q) - \frac{\beta}{2}\left[1 - (p + \gamma q)^2\right]$$

if $p + \gamma q \leq 1$ and $u(p, q) = 0$ otherwise. Solving $u(p^{\max}, q^{\max}) = u_p(p^{\max}, q^{\max}) = 0$ produces $p^{\max} = \beta$ and $\gamma q^{\max} = 1 - \beta$. There is a unique affine solution $Q_\lambda(p) = A + Bp$ of the differential equation (8) that satisfies $Q_\lambda(p^{\max}) = q^{\max}$. The coefficients of the solution are

$$\gamma A = \frac{1}{1 - \beta - \beta\lambda\gamma B},$$
$$\gamma B = \frac{(1-\beta)(1+\lambda) + \sqrt{(1-\beta)^2(1+\lambda)^2 + 4\lambda\beta(2-\beta)}}{2\lambda\beta}.$$

We take $Q_\lambda(p) = 0$ for $p < p^{\text{ask}}$, where p^{ask} is defined by $A + Bp^{\text{ask}} = 0$. It does not matter how Q_λ is defined for $p > p^{\max}$. The candidate equilibrium offer curve is $S^*(p) = Q_\lambda(p)/n$ for $\lambda = (n-1)/n$. One can verify directly that the second-order condition (10) in Theorem 1 is satisfied, so $(S^*, \dots, S^*)$ is a Nash equilibrium.

EXAMPLE 1C: In Example 1, assume $\tilde{\theta}$ is uniformly distributed on $(0, 1)$ and $v(x) = \beta x^2$ for $0 < \beta < 1/2$. For convenience, take $\gamma = 1$. As in the previous example, solving $u = u_p = 0$ produces $p^{\max} = \beta$ and $q^{\max} = 1 - \beta$. This example satisfies all of the assumptions of Biais, Martimort, and Rochet (2000). However, it is a counterexample to Biais, Martimort, and Rochet (2000), because the solution of the differential equation (7) does not define an equilibrium.

To understand why the solution is not an equilibrium, note that $v'(\theta)$ is arbitrarily close to zero when θ is sufficiently small. As a result, the elasticity (16) is not uniformly larger than 1; in other words, there is insufficient adverse selection. Specifically, the conditions in (15) are equivalent, for $p + q < 1$, to

$$q^2 - 2(1-p)q + p^2 - (2 - 1/\beta)p < 0.$$

Given that $\beta < 1/2$, this condition fails at $(p, 0)$ for every $p > 0$. Thus, no matter what n is and no matter what p^{ask} is in $(0, p^{\max})$, the solution of the differential equation starting at $(p^{\text{ask}}, 0)$ and passing through $(p^{\max}, q^{\max})$ must satisfy the hypothesis of Corollary 2 at all sufficiently small $\hat{p} > p^{\text{ask}}$.

Because the hypothesis of Corollary 2 is satisfied for all n, the differential equation does not define an equilibrium for any n. The competitive offer curve in this example is

$$Q(p) = \frac{\sqrt{-3\beta^2 + 12\beta p}}{2\beta} - 1/2 - p$$

with $p^{\max} = \beta$. It is the limit of the solutions Q_λ of (8) as $\lambda \to 1$; however, these solutions are not Nash equilibria for any n. It is an open question whether there are any Nash equilibria in this example that converge to the competitive offer curve.

EXAMPLE 2A: In Example 2, assume $\tilde{v}$ takes only two values, -1 and $+1$, with probability $\frac{1}{2}$ each. Assume that, conditional on $\tilde{q} > 0$, $\tilde{p}$ and $\tilde{q}$ are independent with $\tilde{q}$ being uniformly distributed on $(0, 1)$. Then

$$u(p,q) = -\frac{\phi}{2}(1-p)^+ - \frac{\phi}{2}(-1-p)^+ + \frac{(1-\phi)p}{2}\big(1-G(p)\big)(1-q)^+$$

for all p and all $q \geq 0$, where G is the distribution function of $\tilde{p}$ conditional on $\tilde{q} > 0$ and, as usual, $a^+ = \max(0, a)$. For $0 < q < 1$, u is linear in q, so $u_q(p,q) = u_q(p, nS^*(p))$. This implies

$$\begin{aligned} &u_p(p,q) + (n-1)s^*(p)u_q(p,q) \\ &\quad = u_p(p,q) - u_p\big(p, nS^*(p)\big) \\ &\quad = \big[nS^*(p) - q\big]\left(\frac{1-\phi}{2}\right)\frac{d}{dp}\big\{p\big(1-G(p)\big)\big\}. \end{aligned}$$

By Theorem 2, a necessary condition for S^* to be an equilibrium offer curve is that

$$\frac{d}{dp}\big\{p\big(1-G(p)\big)\big\} \leq 0 \tag{22}$$

whenever $s^*(p) > 0$.

For a more specific example, assume G is the uniform distribution on $(0, 1)$. Solving $u(p^{\max}, q^{\max}) = u_p(p^{\max}, q^{\max}) = 0$ produces $p^{\max} = 1$ and $q^{\max} = (1 - 2\phi)/(1-\phi)$. This is feasible for an equilibrium only when $\phi < 1/2$. Thus, the amount of adverse selection must be limited. For $0 < \lambda \leq 1$, the solution of the differential equation (8) satisfying $Q(p^{\max}) = q^{\max}$ is

$$Q_\lambda(p) = \frac{1}{\lambda(1-\phi)}\int_p^1 \big[2(1-\phi)z - 1\big]\left(\frac{z(1-z)}{p(1-p)}\right)^{1/\lambda}\left(\frac{1}{z(1-z)}\right)dz. \tag{23}$$

Note that (22) is equivalent to $p \geq 1/2$ in this example.

Assume now that $n = 2$. Equation (23) simplifies for $\lambda = 1/2$ to

$$Q(p) = 1 - \frac{\phi}{1-\phi}\left(\frac{1+2p}{3p^2}\right).$$

Solving $Q(p^{\text{ask}}) = 0$ produces

$$p^{\text{ask}} = \frac{\phi + \sqrt{\phi(3-2\phi)}}{3(1-\phi)}.$$

We have $p^{\text{ask}} \geq 1/2$ if and only if $\phi \geq 3/11$. Thus, condition (22) holds for all $p > p^{\text{ask}}$ if and only if $\phi \geq 3/11$. Again, note the need to assume there is a sufficient amount of adverse selection to obtain the second-order condition. When $3/11 < \phi < 1/2$, the second-order condition (10) in Theorem 1 is in fact satisfied for each p, so $S^* = Q/2$ is an equilibrium offer curve.

The minimum probability of an informed trader that implies $p^{\text{ask}} \geq 1/2$ actually increases in n, as shown in Figure 2, from $\phi = 3/11$ for $n = 2$ ($\lambda = 1/2$) to $\phi = 1/3$ for $n = \infty$ ($\lambda = 1$). Thus, if $\phi < 3/11$, then the solution to the differential equation (8) is not an equilibrium offer curve for any n.

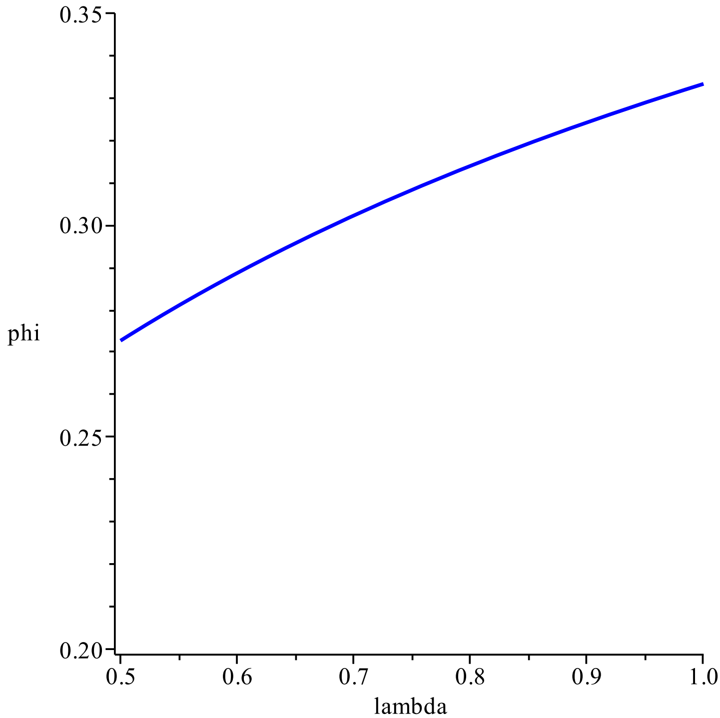

FIGURE 2.—Adverse selection required in Example 2A. If the probability of facing an informed trader is less than the value of ϕ shown, for $\lambda = (n-1)/n$, then the solution of the differential equation (8) is not an equilibrium offer curve. Thus, the required degree of adverse selection is higher when there are more liquidity providers.

EXAMPLE 3A: In Example 3, assume $\tilde{q}$ is exponentially distributed with parameter τ and v is strictly monotone. Then $u_p \geq 0$ everywhere; in particular, $u_p(p, q) \geq 0$ whenever $p < p^{\text{ask}}$. Because (15) holds, the second-order condition (10) in Theorem 1 holds for each p such that $s^*(p) > 0$. Thus, a solution of the differential equation (7) that is strictly increasing above p^{ask} is an equilibrium. For a specific example, take $v(q) = \beta q$. Then there is a unique affine solution $Q_\lambda = A + Bp$ of the differential equation (8), with coefficients $A = -1/(\lambda\tau)$ and $B = 1/\beta$, and $p^{\text{ask}} = \beta/(\lambda\tau)$.

EXAMPLE 3B: In Example 3, assume $\tilde{q}$ is uniformly distributed on $(0, 1)$ and $v(q) = \beta q$. In this case,

$$u(p, q) = p(1 - q) - \frac{\beta}{2}\left(1 - q^2\right)$$

if $q < 1$ and $u(p, q) = 0$ otherwise. Solving $u(p^{\max}, q^{\max}) = u_p(p^{\max}, q^{\max}) = 0$ produces $q^{\max} = 1$. There is a unique affine solution $Q_\lambda(p) = A + Bp$ of the differential equation (8) given by $A = -1/\lambda$ and $B = (1 + \lambda)/(\beta\lambda)$. For every λ, the $p^{\max}$ satisfying $Q_\lambda(p^{\max}) = q^{\max}$ is $p^{\max} = \beta$. Solving $Q_\lambda(p^{\text{ask}}) = 0$ produces $p^{\text{ask}} = \beta/(1 + \lambda)$. As usual, we take $Q_\lambda(p) = 0$ for $p \leq p^{\text{ask}}$ and set $S^* = Q_\lambda/n$ for $\lambda = (n - 1)/n$. In this example, the ratio $u_p(p, q)/u_q(p, q)$ being decreasing in q is equivalent to $p < \beta$, so it holds along the aggregate offer curve Q_λ up to $p = p^{\max}$. However, it is also easy to verify the sufficient condition (10) of Theorem 1 directly, showing that $(S^*, \ldots, S^*)$ is a Nash equilibrium.

7. CONCLUSION

An open question is the nature of an equilibrium, if there is one, when the second-order condition does not hold, as in Example 1C. Because the differential equation is equivalent to pointwise optimality, a natural conjecture is that there may be an equilibrium with bids or offers for discrete quantities where the nondecreasing marginal price constraint is binding. It seems quite likely to us that equilibria of that sort will be in mixed strategies, with random limit prices; otherwise, price cutting would push the outcome to the competitive solution. Our formulation in Assumption 1 does not encompass discrete limit orders from competitors, so a different approach will be required to investigate these issues.

Under standard conditions, the solution Q_λ of (8) converges to the competitive solution as $\lambda \to 1$, equivalently, as $n \to \infty$. When Q_λ is an equilibrium aggregate offer curve, this implies convergence to competition as $n \to \infty$. However, as illustrated in Examples 1C and 2A, the failure of the solution of the differential equation to define an equilibrium when there is insufficient adverse selection is not a "small n" issue. It is an open question whether the competitive solution in such cases is the limit of any sequence of Nash equilibria.

APPENDIX A: PROOFS

PROOF OF THEOREM 1: Let $T \neq T^*$ be a transfer schedule and let P be the price function associated to T. For all $q \geq 0$, we have

$$u\big(P^*(q), q+(n-1)S^*\big(P^*(q)\big)\big) - u\big(P(q), q+(n-1)S^*\big(P(q)\big)\big) \tag{24}$$
$$= \int_{P(q)}^{P^*(q)} \frac{d}{dp} u\big(p, q+(n-1)S^*(p)\big)\, dp$$
$$= \int_{P(q)}^{P^*(q)} \omega\big(p, q+(n-1)S^*(p)\big)\, dp,$$

where ω is defined in (17). We want to show that

$$\omega\big(p, q+(n-1)S^*(p)\big) \begin{cases} \geq 0, & \text{if } p < P^*(q), \\ \leq 0, & \text{if } p > P^*(q). \end{cases} \tag{25}$$

This implies that (24) is nonnegative, so the integral over q is nonnegative, implying that the deviation from T^* to T is unprofitable.

Consider $p > P^*(q)$. Because $P^*(q) = \inf\{y \mid S^*(y) > q\}$, we must have $S^*(p) > q$. This implies $nS^*(p) > q + (n-1)S^*(p)$. From (10), this implies $\omega(p, q+(n-1)S^*(p)) \leq 0$.

Now, consider $p < P^*(q)$. Assume P^* is continuous at q. Then there exists $x < q$ such that $p < P^*(x)$. Because $P^*(x) = \inf\{y \mid S^*(y) > x\}$, this implies $S^*(p) \leq x < q$. Thus, $nS^*(p) < q + (n-1)S^*(p)$ and (10) implies $\omega(p, q + (n-1)S^*(p)) \geq 0$.

The set of points at which P^* is discontinuous is at most countable, so we conclude that, for almost all q, (25) holds for all p. Thus, integrating (24) over q shows that P^* is an optimal response. *Q.E.D.*

PROOF OF COROLLARY 1: We want to verify (10). Define ω by (17). Consider any $q > nS^*(p)$. By assumption (a), $\omega(p, q) \geq 0$ if $u_q(p, q) \geq 0$ or if $s^*(p) = 0$. If $u_q(p, q) < 0$ and $s^*(p) \neq 0$, then assumption (e) implies $\omega(p, q) \geq 0$. Now, consider any $q < nS^*(p)$. By assumption (b), $\omega(p, q) \leq 0$ if $u_p(p, q) \leq 0$. Assume $u_p(p, q) > 0$. By assumption (d), $s^*(p) > 0$, and by assumptions (b) and (c), $u_q(p, q) < 0$. Therefore, assumption (e) implies $\omega(p, q) \leq 0$. *Q.E.D.*

PROOF OF THEOREM 2: Assume (13a) holds. Define ω by (17). There exists an interval (p_1, p_2) containing $\hat{p}$ and $\epsilon > 0$ such that $s^*(p) > \epsilon$ and $\omega(p, q) < 0$

whenever $p_1 \leq p \leq p_2$ and $nS^*(p) < q < nS^*(p) + \epsilon$. Define $\hat{s}(p) = s^*(p)$ for $p \notin (p_1, p_2)$ and

$$\hat{s}(p) = \begin{cases} s^*(p) + \epsilon, & \text{if } p_1 \leq p < (p_1 + p_2)/2, \\ s^*(p) - \epsilon, & \text{if } (p_1 + p_2)/2 \leq p \leq p_2 . \end{cases}$$

Define $\hat{S}(p) = \int_{-\infty}^{p} \hat{s}(x)\,dx$. We will show that $\hat{S}$ is a profitable deviation. Set $a = S^*(p_1)$ and $b = S^*(p_2)$, and set $\hat{P}(q) = \inf\{p \mid \hat{S}(p) > q\}$. Note that $\hat{S}(p) > S^*(p)$ for $p_1 < p < p_2$, so $\hat{P}(q) < P^*(q)$ for $a < q < b$. The expected gain from the deviation is

$$\begin{aligned} &\int_a^b \big[u\big(\hat{P}(q), q + S^*_{-i}\big(\hat{P}(q)\big)\big) - u\big(P^*(q), q + S^*_{-i}\big(P^*(q)\big)\big)\big]\,dq \\ &\quad = \int_a^b \int_{P^*(q)}^{\hat{P}(q)} \omega\big(p, q + S^*_{-i}(p)\big)\,dp\,dq \\ &\quad = -\int_a^b \int_{\hat{P}(q)}^{P^*(q)} \omega\big(p, q + S^*_{-i}(p)\big)\,dp\,dq. \end{aligned}$$

Note that $a < q < b$ implies $p_1 \leq \hat{P}(q) \leq P^*(q) \leq p_2$. Also, $\hat{P}(q) < p < P^*(q)$ implies $S^*(p) \leq q \leq \hat{S}(p)$. Thus, on the domain of integration, $p_1 \leq p \leq p_2$, $S^*(p) \leq q \leq \hat{S}(p) \leq S^*(p) + \epsilon$, and $\omega(p, q + S^*_{-i}(p)) < 0$. When (13b) holds, we can likewise define a profitable deviation via

$$\hat{s}(p) = \begin{cases} s^*(p) - \epsilon, & \text{if } p_1 \leq p < (p_1 + p_2)/2, \\ s^*(p) + \epsilon, & \text{if } (p_1 + p_2)/2 \leq p \leq p_2. \end{cases}$$

Q.E.D.

PROOF OF COROLLARY 2: Under these assumptions,

$$\frac{u_p(p,q)}{u_q(p,q)} > \frac{u_p(p, nS^*(p))}{u_q(p, nS^*(p))} = -(n-1)s^*(p)$$

for $q > nS^*(p)$ on a neighborhood of $(\hat{p}, nS^*(\hat{p}))$. This implies

$$u_p(p,q) + (n-1)s^*(p)u_q(p,q) < 0,$$

so (13a) holds. *Q.E.D.*

PROOF OF LEMMA 1: Let F and f denote the distribution and density function of a normal (μ, σ^2) random variable, and set $R(x) = (1 - F(x))/f(x)$. We have

$$\frac{f'(x)}{f(x)} = -\frac{x - \mu}{\sigma^2}. \tag{26}$$

Using this fact and applying l'Hôpital's rule to $(x-\mu)R(x)$ as $(x-\mu)(1-F(x))/f(x)$, we obtain $\lim_{x\to\infty}(x-\mu)R(x)=\sigma^2$. Also

$$R'(x)=-1+\left(\frac{x-\mu}{\sigma^2}\right)R(x), \tag{27a}$$

$$R''(x)=-\frac{x-\mu}{\sigma^2}+\left(\frac{x-\mu}{\sigma^2}\right)^2 R(x)+\frac{1}{\sigma^2}R(x). \tag{27b}$$

Now we show that $R'<0$. This is equivalent to

$$\left(\frac{x-\mu}{\sigma^2}\right)\big(1-F(x)\big)-f(x)<0. \tag{28}$$

The left-hand side of (28) is increasing in x, because its derivative is $(1-F(x))/\sigma^2>0$. It tends to zero as $x\to\infty$, so it must be negative for all finite x.

The formulas (27) imply that $R''>0$ if and only if (20) holds. So it remains to show that R is convex, which is equivalent to

$$-\left(\frac{x-\mu}{\sigma^2}\right)f(x)+\left(\frac{x-\mu}{\sigma^2}\right)^2\big(1-F(x)\big)+\frac{1}{\sigma^2}\big(1-F(x)\big)>0. \tag{29}$$

The left-hand side of (29) is decreasing in x. In fact, the derivative is

$$\begin{aligned}\frac{2(x-\mu)}{\sigma^4}\big(1-F(x)\big)-\frac{2}{\sigma^2}f(x)&=\frac{2}{\sigma^2}\left(\frac{x-\mu}{\sigma^2}\int_x^\infty f(y)\,dy-f(x)\right)\\&<\frac{2}{\sigma^2}\left(\int_x^\infty\frac{y-\mu}{\sigma^2}f(y)\,dy-f(x)\right)\\&=\frac{2}{\sigma^2}\left(-\int_x^\infty f'(y)\,dy-f(x)\right)=0.\end{aligned}$$

The left-hand side of (29) converges to zero as $x\to\infty$, so it must be positive for all finite x.

Finally, note that

$$R(\bar{v}+\sigma)=\frac{1-F^*(1)}{f^*(1)}\times\sigma=0.6556795\times\sigma,$$

where F^* and f^* denote the standard normal distribution and density functions. Thus, $R(\bar{v}+\sigma)>\sigma/2$. *Q.E.D.*

PROOF OF LEMMA 2:

Step 1: $\lambda=1$. First, we show that there exists p_c such that $u(p_c,0)=0$. This is equivalent to $(p_c-\bar{v})R(p_c)=\beta\sigma^2$. The function $G(p)\equiv(p-\bar{v})R(p)$ is an increasing function of p, starting at 0 at $p = \bar{v}$ and converging to σ^2 as $p \to \infty$. To see that it is increasing, differentiate and use (20). To see the limit as $p \to \infty$, apply l'Hôpital's rule to the ratio $(p - \bar{v})(1 - F(p))/f(p)$, using (26). Thus, there exists a unique p_c such that $u(p_c, 0) = 0$.

Now consider any $p > p_c$. We will show that there is a unique solution $q = Q_c(p)$ to $u(p, q) = 0$. The function R is decreasing and $\lim_{x\to\infty} R(x) = 0$. Because G is increasing,

$$(p - \bar{v})R(p) > (p_c - \bar{v})R(p_c) = \beta\sigma^2,$$

so $R(p) > \beta\sigma^2/(p - \bar{v})$. Therefore, there is a unique $Q_c(p)$ such that $R(p + \gamma Q_c(p)) = \beta\sigma^2/(p - \bar{v})$; equivalently, $u(p, Q_c(p)) = 0$. Differentiating $(p - \bar{v})R(p + \gamma Q_c(p)) = \beta\sigma^2$ yields

$$(p - \bar{v})R'\big(p + \gamma Q_c(p)\big)\big[1 + \gamma Q_c'(p)\big] + R\big(p + \gamma Q_c(p)\big) = 0.$$

This implies

$$1 + \gamma Q_c'(p) = -\frac{R(p + \gamma Q_c(p))}{R'(p + \gamma Q_c(p))} \cdot \frac{1}{p - \bar{v}} > \frac{p + \gamma Q_c(p) - \bar{v}}{p - \bar{v}} > 1,$$

using Lemma 1. Therefore, $Q_c' > 0$. The condition $u(p, Q_c(p)) = 0$ implies that Q_c solves the differential equation (8) for $\lambda = 1$. Because

$$R\big(p + \gamma Q_c(p)\big) = \frac{\beta\sigma^2}{p - \bar{v}} \to \frac{\beta\sigma^2}{p_c - \bar{v}} = R(p_c)$$

as $p \downarrow p_c$, we have $\lim_{p\downarrow p_c} Q_c(p) = 0$. Therefore, we can extend Q_c continuously by setting $Q_c(p) = 0$ for all $p \le p_c$.

Step 2: $\lambda = 0$. By differentiating (19), we see that the condition $u_p(p, q) = 0$ is equivalent to $v(x) - p + R(x) = 0$. First, we show that there exists p_m such that $u_p(p_m, 0) = 0$. The function $K(p) \equiv v(p) - p + R(p)$ is decreasing, positive at $p = \bar{v} + \sigma$, and converges to $-\infty$ as $p \to \infty$. To see that it is positive at $p = \bar{v} + \sigma$, note that

$$v(\bar{v} + \sigma) - \bar{v} - \sigma + R(\bar{v} + \sigma) = (\beta - 1)\sigma + R(\bar{v} + \sigma) > 0,$$

owing to Lemma 1 and the assumption $\beta > 1/2$. To see that K is decreasing with $\lim_{p\to\infty} K(p) = -\infty$, note that $K'(p) = \beta - 1 + R'(p) < \beta - 1 < 0$. It follows that there is a unique p_m such that $K(p_m) = 0$, equivalently, $u_p(p_m, 0) = 0$. Moreover, $p_m > \bar{v} + \sigma$.

Now, consider any $p > p_m$. We will show that there is a unique solution $q = Q_m(p)$ to $u_p(p, q) = 0$. Set $L(x) = v(x) + R(x)$. Because $K' < \beta - 1$, we have $L' < \beta$. The function L inherits the convexity of R, so for any $x > p_m$,

$$L'(x) \ge L'(\bar{v} + \sigma) = \beta + R'(\bar{v} + \sigma) = \beta - 1 + \frac{1}{\sigma}R(\bar{v} + \sigma),$$

using (27a) for the second equality. Using the assumption $\beta > 1/2$ and Lemma 1, we obtain $L'(x) > 0$. The strictly positive lower bound on L' implies $\lim_{x\to\infty} L(x) = \infty$. Finally, because $L(p_m) = p_m$, $p > p_m$, and $L' < 1$, we have $L(p) < p$. Thus, there is a unique $Q_m(p)$ satisfying $L(p + \gamma Q_m(p)) = p$; equivalently, $u_p(p, Q_m(p)) = 0$. Differentiating $L(p+\gamma Q_m(p)) = p$ and using $L' < \beta$ implies

$$Q_m'(p) > \frac{1-\beta}{\gamma\beta} > 0. \tag{30}$$

The condition $u_p(p, Q_m(p)) = 0$ is equation (8) for $\lambda = 0$. Because

$$v\big(p + \gamma Q_m(p)\big) + R\big(p + \gamma Q_m(p)\big) = p \to p_m = v(p_m) + R(p_m)$$

as $p \downarrow p_m$, we have $\lim_{p\downarrow p_m} Q_m(p) = 0$. Therefore, we can extend Q_m continuously by setting $Q_m(p) = 0$ for $p \le p_m$.

Step 3: $Q_c > Q_m$. We have

$$p = v\big(p + \gamma\sigma Q_m(p)\big) + R\big(p + \gamma\sigma Q_m(p)\big)$$

for $p \ge p_m$ and

$$p = \bar{v} + \frac{\beta\sigma^2}{R(p + \gamma Q_c(p))}$$

for $p \ge p_c$. We will show that

$$v(x) + R(x) > \bar{v} + \frac{\beta\sigma^2}{R(x)} \tag{31}$$

for all $x > \bar{v}$. Both sides of this inequality are increasing in x (see Step 2 for $v+R$). It follows that we must have $p_m > p_c$ and $Q_c(p) > Q_m(p)$ for all $p > p_c$.

The inequality (31) is equivalent to $J(x) > 0$, where we define

$$J(x) = \beta(x - \bar{v})R(x) + R(x)^2 - \beta\sigma^2.$$

Lemma 1 implies that $\lim_{x\to\infty} J(x) = 0$. Using the formula (27a), we can calculate

$$J'(x) = \frac{x-\bar{v}}{\sigma^2}J(x) + \left[\frac{x-\bar{v}}{\sigma^2}R(x) + \beta - 2\right]R(x).$$

Also,

$$\begin{aligned}\left[\frac{x-\bar{v}}{\sigma^2}R(x) + \beta - 2\right]R(x) &= \big[\beta(x-\bar{v})R(x) + \beta(\beta-2)\sigma^2\big]\frac{R(x)}{\beta\sigma^2} \\ &< J(x)\frac{R(x)}{\beta\sigma^2},\end{aligned}$$

using $\beta < 1$ and $R(x)^2 > 0$ for the inequality. Hence

$$J'(x) < \left[\frac{x-\bar{v}}{\sigma^2} + \frac{R(x)}{\beta\sigma^2}\right] J(x).$$

Thus, if $J(x) \leq 0$ for any $x > \bar{v}$, then $J'(x) < 0$. This is inconsistent with $\lim_{x\to\infty} J(x) = 0$, so we conclude that $J(x) > 0$ for all $x > \bar{v}$.

Step 4: $0 < \lambda < 1$. Fix $\lambda \in (0, 1)$. We want to solve

$$\frac{dQ_\lambda(p)}{dp} = \psi_\lambda\big(p, Q_\lambda(p)\big), \tag{32}$$

where

$$\psi_\lambda(p, q) = -\frac{u_p(p, q)}{\lambda u_q(p, q)} = -\frac{1}{\lambda\gamma} + \frac{R(p+\gamma q)}{\lambda\gamma[p - v(p+\gamma q)]}.$$

For each positive integer $k > p_m$, consider the region

$$D_k = \big\{(p, q) \mid p_c \leq p \leq k, Q_m(p) \leq q \leq Q_c(p)\big\}.$$

Note that $(p, 0) \in D_k$ for each $p \in [p_c, p_m]$. For $q < Q_c(p)$, we have, from the monotonicity of v,

$$\begin{aligned} p - v(p+\gamma q) &= \bar{v} + \frac{\beta\sigma^2}{R(p+\gamma Q_c(p))} - v(p+\gamma q) \\ &> \bar{v} + \frac{\beta\sigma^2}{R(p+\gamma Q_c(p))} - v\big(p+\gamma Q_c(p)\big) > 0. \end{aligned} \tag{33}$$

This implies that ψ_λ is continuously differentiable in q on D_k. Also, for $q > Q_m(p)$, we have, due to the monotonicity of the function L defined in Step 2,

$$R(p+\gamma q) > p - v(p+\gamma q). \tag{34}$$

Combining (33) and (34) shows that $\psi_\lambda(p, q) \geq 0$ on D_k with equality only at $q = Q_m(p)$. Moreover, $\psi_\eta(p, q)$ is a decreasing function of η for $(p, q) \in D_k$.

Because ψ_λ is continuously differentiable in q on D_k, it satisfies a local Lipschitz condition, which implies that, for each initial condition $a \in (p_c, p_m)$, there exists a solution of (32) satisfying $Q_\lambda(a) = 0$. The solution is defined up to the price p where $Q_\lambda(p)$ hits the boundary of D_k.

Let Π_c denote the set of $p \geq p_c$ such that the solution of (32) with initial condition $Q_\lambda(p) = 0$ exits the region D_k at the boundary $q = Q_c(p)$. Because ψ_η is decreasing in η, the solution of (32) with initial condition $Q_\lambda(p_c) = 0$ is larger than Q_1, by a standard comparison theorem (e.g., Cole (1968, Theorem 9-2.1)). Thus, $p_c \in \Pi_c$. The comparison theorem also implies that, for any

$p \in \Pi_c$, the interval $[p_c, p]$ is contained in Π_c. Thus, Π_c is an interval $[p_c, \hat{p}]$. The interval is closed on the right by the continuous dependence of solutions on initial conditions.

Likewise, let Π_m denote the set of $p \leq p_m$ such that the solution of (32) with initial condition $Q_\lambda(p) = 0$ exits the region D_k at the boundary $q = Q_m(p)$. Because ψ_η is decreasing in η, the solution of (32) with initial condition $Q_\lambda(p_m) = 0$ is smaller than Q_0, by the comparison theorem, so $p_m \in \Pi_m$. Again, the comparison theorem and the continuous dependence of solutions on initial conditions imply that Π_m is an interval $[\check{p}, p_m]$.

The two intervals $[p_c, \hat{p}]$ and $[\check{p}, p_m]$ are disjoint, because the boundaries $q = Q_c(p)$ and $q = Q_m(p)$ do not intersect. Therefore, the interval $(\hat{p}, \check{p})$ is nonempty. Choose any $p_k \in (\hat{p}, \check{p})$. By definition, the solution of (32) with initial condition $Q_\lambda(p_k) = 0$ does not exit D_k at either of the boundaries $q = Q_c(p)$ or $q = Q_m(p)$, so it must exit at the boundary $p = k$. There exists a subsequence of the p_k that has a limit $p_\lambda \in [p_m, p_c]$. By the continuous dependence of solutions on initial conditions, the solution Q_λ of (32) with initial condition $Q_\lambda(p_\lambda) = 0$ does not exit any D_k at either of the boundaries $q = Q_c(p)$ or $q = Q_m(p)$, so it satisfies $Q_m(p) \leq Q_\lambda(p) \leq Q_c(p)$ for all $p < \infty$. *Q.E.D.*

PROOF OF THEOREM 3: To verify (10), we consider three regions: (i) $p \leq p^{\text{ask}}$, (ii) $v(x) \geq p > p^{\text{ask}}$, and (iii) $p > v(x)$ and $p > p^{\text{ask}}$. Define ω by (17).

In region (i), we have $q > nS^*(p)$ and $s^*(p) = 0$, so we need to show that $u_p(p, q) \geq 0$. For any $q > 0$, there exists $p' > p^{\text{ask}} \geq p$ such that $u_p(p', q) = 0$. To show that $u_p(p, q) \geq 0$, equivalently, $u_p(p, q)/f(p, q) \geq 0$, it suffices to show that u_p/f is decreasing in p. This is equivalent to $R' + v' - 1 < 0$, and we have both $R' < 0$ and $v' < 1$, so $\omega(p, q) \geq 0$ in region (i).

In region (ii), we have $u(p, q) \leq 0$, so we are to the right (in the usual qp plane) of nS^*. We need to show here that $\omega(p, q) \geq 0$. This is true, because we have both $u_q(p, q) \geq 0$ and $u_p(p, q) > 0$.

In region (iii), we will verify the equivalent conditions in (15), which implies (10). We rewrite the second condition in (15), using $R' < 0$ and $v(x) = (1 - \beta)\bar{v} + \beta x$, as

$$(35) \qquad (1 - \beta)\bar{v} + \beta x - p - \beta \frac{R(x)}{R'(x)} > 0.$$

The derivative of the left-hand side of (35) with respect to q is

$$\frac{\gamma\beta R(x) R''(x)}{R'(x)^2} > 0,$$

due to the convexity of R. Thus, the left-hand side of (35) is an increasing function of q, and it is positive for all q if it is positive at $q = 0$. Hence, we need to show that

$$-(1-\beta)(p-\bar{v}) - \beta\frac{R(p)}{R'(p)} > 0.$$

From (20), we have

$$-(1-\beta)(p-\bar{v}) - \beta\frac{R(p)}{R'(p)} > -(1-\beta)(p-\bar{v}) + \beta(p-\bar{v}).$$

This is positive, because $\beta > 1/2$ and $p > \bar{v}$. *Q.E.D.*

APPENDIX B: EQUIVALENCE OF THE DIFFERENTIAL EQUATIONS

The purpose of this appendix is to demonstrate that the differential equation (43) and boundary condition derived by Biais, Martimort, and Rochet (2000) are equivalent, in the case of Example 1, to the differential equation and boundary conditions studied in this paper. Biais, Martimort, and Rochet (2000) assumed the type θ of the CARA investor has bounded support $[\underline{\theta}, \bar{\theta}]$, and they characterized equilibrium in terms of the quantity $Q(\theta)$ received by the CARA investor when his type is θ. The differential equation (43) is

$$\text{(36a)} \qquad \forall \theta \in (\theta_a, \bar{\theta}), \quad Q'(\theta) = \frac{1}{\gamma}\left(1 + \frac{(n-1)(q^*(\theta) - Q(\theta))}{n(Q(\theta) - q_m(\theta))}\right)^{-1},$$

where $\theta_a = \inf\{\theta \mid Q(\theta) > 0\}$, and q^* and q_m are defined as

$$\text{(36b)} \qquad q^*(\theta) = \frac{\theta - v(\theta)}{\gamma},$$

$$\text{(36c)} \qquad q_m(\theta) = q^*(\theta) - \frac{R(\theta)}{\gamma}.$$

Proposition 8 of Biais, Martimort, and Rochet (2000) states that

$$\text{(36d)} \qquad \forall \theta \in (\theta_a, \bar{\theta}), \quad q_m(\theta) < Q(\theta) < q^*(\theta).$$

Also, Proposition 8 states that Q is strictly increasing on the domain $[\theta_a, \bar{\theta}]$. Finally, the boundary condition stated in Proposition 7 is

$$\text{(36e)} \qquad Q(\bar{\theta}) = q^*(\bar{\theta}).$$

Here, we are writing Q for the transaction quantity denoted as q^n by Biais, Martimort, and Rochet (2000). Also, $v(\theta) = \mathsf{E}[\tilde{v} \mid \tilde{\theta} = \theta]$, γ is the product of the risk aversion of the CARA investor with the variance of the assumed to be normally distributed random variable $\tilde{v} - \mathsf{E}[\tilde{v} \mid \tilde{\theta}]$, and $R(\theta) = (1 - F(\theta))/f(\theta)$, where F is the distribution function of $\tilde{\theta}$ and f is the density function of $\tilde{\theta}$. Using (36a) and (36d) and the first-order condition for the CARA investor, it can be shown that the aggregate transfer schedule T must be twice differentiable and strictly convex on the domain $(0, Q(\bar{\theta}))$. This implies that the associated price function $P = T'$ is differentiable and strictly increasing on the same domain. In our notation, the inverse of the aggregate price function P is nS^*, where S^* is the offer curve of each trader. Thus,

$$n\frac{dS^*(p)}{dp} = \frac{1}{P'(nS^*(p))}$$

on $\{p \mid 0 < nS^*(p) < Q(\bar{\theta})\}$.

Because Q restricted to $[\theta_a, \bar{\theta}]$ is strictly increasing, it is invertible on its range $[0, Q(\bar{\theta})]$. Denote the inverse function by $q \mapsto \theta(q)$. From (36a), substituting (36b) and (36c), we obtain

$$\begin{aligned}\theta'(q) &= \gamma\left(1 + \frac{(n-1)(q^*(\theta(q)) - q)}{n(q - q_m(\theta(q)))}\right) \\ &= \gamma\left(1 + \frac{n-1}{n} \cdot \frac{\theta(q) - v(\theta(q)) - \gamma q}{\gamma q + R(\theta(q)) + v(\theta(q)) - \theta(q)}\right).\end{aligned}$$

The first-order condition for the CARA investor is $\theta(q) - \gamma q = P(q)$, so

$$P'(q) = \theta'(q) - \gamma = \gamma\left(\frac{n-1}{n}\right)\left(\frac{\theta(q) - v(\theta(q)) - \gamma q}{\gamma q + R(\theta(q)) + v(\theta(q)) - \theta(q)}\right)$$

for $q \in (0, Q(\bar{\theta}))$. It follows that

$$\frac{dS^*(p)}{dp} = \frac{1}{\gamma(n-1)}\left(\frac{\gamma q + R(\theta(q)) + v(\theta(q)) - \theta(q)}{\theta(q) - v(\theta(q)) - \gamma q}\right)$$

on $\{p \mid 0 < nS^*(p) < Q(\bar{\theta})\}$, where q on the right-hand side is $nS^*(p)$. Substituting $\theta(q) = P(q) + \gamma q = p + \gamma nS^*(p)$ gives

$$\frac{dS^*(p)}{dp} = \frac{1}{\gamma(n-1)}\left(\frac{R(p + \gamma nS^*(p)) + v(p + \gamma nS^*(p)) - p}{p - v(p + \gamma nS^*(p))}\right),$$

which we can rearrange as

$$a\big(p, nS^*(p)\big) + (n-1)b\big(p, nS^*(p)\big)\frac{dS^*(p)}{dp} = 0, \tag{37a}$$

where

$$\text{(37b)} \qquad a(p,q) = R(p+\gamma q) + v(p+\gamma q) - p,$$

$$\text{(37c)} \qquad b(p,q) = \gamma\big[v(p+\gamma q) - p\big].$$

Now consider the differential equation (7) in the CARA case. From (2), we have

$$u_p(p,q) = a(p,q)f(p+\gamma q),$$
$$u_q(p,q) = b(p,q)f(p+\gamma q).$$

Hence, the differential equation (7) is equivalent to (37).

Section 6 uses the notation $q^{\max}$ for the maximum quantity transacted in equilibrium and $p^{\max}$ for the maximum price. The boundary condition (36e) states that $q^{\max} = (\bar{\theta} - v(\bar{\theta}))/\gamma$. From the first-order condition $p = \theta - \gamma q$, we obtain that $p^{\max} = v(\bar{\theta})$. It follows immediately from (2) that $u(p^{\max}, q^{\max}) = 0$. Moreover, computing the derivative from the left, we have

$$\begin{aligned} u_p(p^{\max}, q^{\max}) &= \big[v(p^{\max} + \gamma q^{\max}) - p^{\max}\big]f\big(p^{\max} + \gamma q^{\max}\big) \\ &\quad + 1 - F\big(p^{\max} + \gamma q^{\max}\big) \\ &= \big[v(\bar{\theta}) - v(\bar{\theta})\big]f(\bar{\theta}) + 1 - F(\bar{\theta}) = 0. \end{aligned}$$

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