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Increasing risk aversion and life-cycle investing

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Abstract
We derive the optimal portfolio for an investor with increasing relative risk aversion in a complete continuous-time securities market. The IRRA assumption helps to mitigate the criticism of constant relative risk aversion that it implies an unreasonably large aversion to large gambles, given reasonable aversion to small gambles. The model provides theoretical support for the common recommendation of financial advisors that older investors should reduce their allocations to risky assets, and it is consistent with empirical relations between age, wealth, and portfolios.

Keywords Risk aversion · Portfolio choice · Life-cycle investment

JEL Classification G11

We demonstrate two useful features of increasing relative risk aversion (IRRA) preferences. The first has to do with aversion to small versus large gambles. Kandel and Stambaugh [10] observe that a calibration of constant relative risk aversion (CRRA) preferences that seems reasonable for small gambles implies unreasonably large aversion to large gambles. Conversely, a calibration to large gambles implies unreasonably small aversion to small gambles. This is an undesirable feature of any model based on CRRA preferences. Models based on IRRA preferences are less subject to this concern, because the premium an investor would pay to avoid a gamble does not grow as fast with the size of the gamble for IRRA preferences as it does for CRRA preferences.[1]

[1] IRRA preferences do not eliminate the concern entirely. Rabin [15] argues that it is a feature of expected utility in general that either aversion to small gambles is too small, or aversion to large gambles is too large.

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The second useful feature, and the main focus of this paper, is related to the standard advice of financial planners that investors should reduce their risk exposures as they age [2,5]. This has always been regarded as a bit of a puzzle, because with CRRA utility and a constant investment opportunity set, the optimal risk exposure is constant over time [16]. However, this is not true for IRRA investors. An investor with IRRA preferences who has experienced good returns and can consequently forecast with confidence a high future level of consumption becomes less willing to gamble and will shift her money to less risky assets. We call this a lock-in effect. The extent to which the lock-in effect occurs depends on the history of returns. We investigate this by Monte Carlo analysis. Our simulations show that an IRRA investor will typically reduce her risk exposure as she ages, consistent with the advice of financial planners.[2]

Numerous papers have examined various issues related to the portfolio choice problem, including borrowing constraints, employment flexibility, retirement flexibility, and stochastic investment opportunity sets, to attempt to rationalize the advice to decrease portfolio risk with age (see [1], for a survey). We do not claim that the other issues examined previously are unimportant or are less important than IRRA. Our contribution is simply to show that IRRA can also contribute to motivating the advice. Our simulations show that IRRA can have a substantial effect on optimal portfolios.

The utility functions that we study are of the form

$$u(w) = \frac{1}{1-\rho}(\xi + w)^{1-\rho} \tag{1}$$

for constant parameters $\rho, \xi > 0$. These are the only utility functions that have hyperbolic absolute risk aversion (HARA) and that have both decreasing absolute risk aversion (DARA) and IRRA.[3] The relative risk aversion is

$$\frac{w\rho}{\xi + w}, \tag{2}$$

which increases from 0 to ρ as w increases from 0 to ∞.

The entire HARA class is studied by Merton [13], but Merton's analysis does not impose nonnegativity of consumption. For utility functions of the form (1), Merton's solution allows for negative consumption and negative wealth with positive probability. Previous authors who note this issue and attempt explicit solutions impose a boundary condition on the value function at $W_t = 0$ [17,18]. We take the more direct approach of requiring consumption to be nonnegative.

The plan of the paper is as follows. Section 1 explains how the aversion to small versus large gambles is different for IRRA investors than for CRRA investors. Section 2 presents and solves the portfolio choice problem. Section 3 uses Monte Carlo analysis to show how the optimal portfolio changes with age for an IRRA investor. Section 4 shows that IRRA is consistent with the empirically observed positive cross-sectional correlation between wealth and portfolio risk. There is a literature that examines how labor income risk affects the

2 The empirical relation of equity allocations to age is not entirely clear. Bodie and Crane [2] find that the allocation is decreasing in age. Also, Curcuru et al. [7] find that the allocation is decreasing in age for low-wealth individuals and hump-shaped for others. However, Ameriks and Zeldes [1] find that the allocation is increasing in age for individual investors, with the hump-shaped cross-sectional pattern being due to cohort effects.

3 There are different results in the literature regarding whether these utility functions are actually representative of individuals' preferences. [3] argue that they are the most representative within the HARA class, based on experimental evidence. However, [4] argue that decreasing relative risk aversion is a better assumption, based on Swedish household portfolio data.

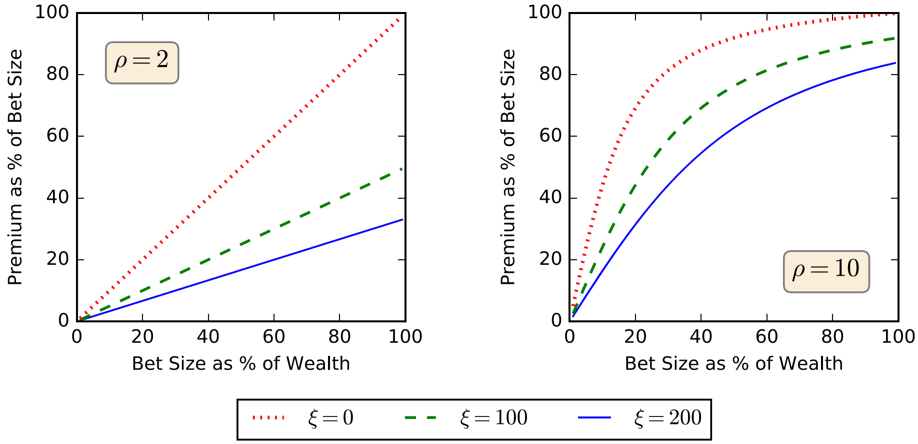

Fig. 1 Aversion to small and large gambles. The premium-to-bet ratio p/b is shown as a function of the bet-to-wealth ratio b/w, when the utility function is (1) and the premium is p defined by (3). The dotted line is for CRRA preferences. We normalize $w = 100$. The dashed and solid lines are for IRRA preferences, with $\xi = 100$ and $\xi = 200$ respectively.

optimal dependence of portfolio risk on age. Section 5 explains how our results relate to that literature. Section 6 concludes the paper.

1 Risk aversion in the small and in the large

Consider a 50–50 gamble of size b. The premium p that an investor with utility function u would pay to avoid the gamble is the solution of the equation

$$u(w - p) = \frac{1}{2}u(w - b) + \frac{1}{2}u(w + b). \tag{3}$$

The observation of Kandel and Stambaugh ([10], p. 68) that aversion to large gambles seems too large when aversion to small gambles is reasonable is a consequence of the fact that the premium-to-bet ratio p/b increases when the bet-to-wealth ratio b/w increases. For example, with CRRA equal to 4, which is generally regarded as reasonable, an investor with wealth of \$100,000 would pay approximately \$20 to avoid a 50–50 bet for \$1000 but would pay nearly \$75,000 to avoid a 50–50 bet for \$80,000.

The premium-to-bet ratio increases less rapidly for IRRA investors than for CRRA investors with the same value of ρ. For example, when $\rho = 2$, the premium-to-bet ratio is linear in the bet-to-wealth ratio with a coefficient that is a decreasing function of the degree of IRRA. Specifically,

$$\frac{p}{b} = \frac{w}{w + \xi} \cdot \frac{b}{w}.$$

Figure 1 illustrates the fact that aversion to gambles grows more slowly for IRRA preferences than for CRRA preferences for $\rho = 2$ and $\rho = 10$. The same is true for any value of ρ.

2 Portfolio choice with increasing risk aversion

The main topic of this paper is life-cycle portfolio choice with IRRA preferences. We use standard techniques to solve the portfolio choice problem, following Karatzas, Lehoczky and Shreve [11] and Cox and Huang [6]. Then, we use Monte Carlo analysis to investigate properties of the solution.

Assume there is a constant risk-free rate r and a single risky asset with total return

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B \tag{4}$$

for constants μ and σ, where B is a standard Brownian motion. The assumption that there is a single risky asset is without loss of generality, given that we assume a constant investment opportunity set, because all expected utility investors hold combinations of the risk-free asset and the tangency portfolio when the investment opportunity set is constant [14]. Under these assumptions, the stochastic discount factor (SDF) process M is defined by $M_0 = 1$ and

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B, \tag{5}$$

where $\lambda = (\mu - r)/\sigma$ is the Sharpe ratio of the risky asset.

This asset market is dynamically complete. We assume there are no constraints against borrowing or short sales. The only constraints are that consumption and terminal wealth must be nonnegative. Let W denote the investor's wealth, with W_0 given exogenously. The wealth W_0 includes the value of any future labor income plus the value of assets owned. Any nonnegative adapted process C is a feasible consumption process for the investor provided it satisfies the budget constraint

$$\mathsf{E}\int_0^T M_t C_t \le W_0. \tag{6}$$

The investor chooses a consumption process $C \ge 0$ to solve:

$$\max \quad \mathsf{E}\int_0^T \frac{\mathrm{e}^{-\delta t}}{1-\rho}(\xi + C_t)^{1-\rho}\,\mathrm{d}t \quad \text{subject to} \quad \mathsf{E}\int_0^T M_t C_t\,\mathrm{d}t \le W_0. \tag{7}$$

The optimization problem (7) is mathematically equivalent to maximizing CRRA utility of consumption with a positive lower bound on consumption. Specifically, if we set $D_t = \xi + C_t$, then the problem (7) is equivalent to

$$\max_D \quad \mathsf{E}\int_0^T \frac{\mathrm{e}^{-\delta t}}{1-\rho} D_t^{1-\rho}\,\mathrm{d}t \quad \text{subject to} \quad \mathsf{E}\int_0^T M_t D_t\,\mathrm{d}t \le \hat{W}_0 \quad \text{and} \quad D \ge \xi, \tag{8}$$

where

$$\hat{W}_0 = W_0 + \xi\mathsf{E}\int_0^T M_t\,\mathrm{d}t. \tag{9}$$

The problem (8) is studied by Lakner and Nygren [12] and Shin et al. [19]. Lakner and Nygren characterize optimal portfolios in terms of the Malliavin calculus for general utility functions, whereas we compute explicit formulas for IRRA/CRRA utility. Shin et al. provide explicit formulas for CRRA utility. However, we cannot apply their results directly, because we assume a finite horizon and they assume an infinite horizon, making their problem stationary. Furthermore, the transformation of our problem to theirs requires the redefinition of wealth illustrated in (9), which leads to different economic results. For example, the allocation to risky assets is a decreasing function of wealth in our model (see Fig. 2), but it is an increasing function of wealth when wealth is redefined as in (9) (see Shin et al.'s Fig. 2).

Returning to the optimization problem (7), the Kuhn–Tucker condition is, for a Lagrange multiplier η,

$$\mathrm{e}^{-\delta t}(\xi + C_t)^{-\rho} \le \eta M_t$$

with equality when $C_t > 0$. Define $\gamma = \eta^{-1/\rho}$ and

$$X_t = \left(\mathrm{e}^{\delta t} M_t\right)^{-1/\rho} . \tag{10}$$

Then, the Kuhn–Tucker condition can be expressed as:

$$C_t = (\gamma X_t - \xi)^+ . \tag{11}$$

The optimal wealth process is W defined by

$$\begin{aligned} W_t &= \frac{1}{M_t} \mathsf{E}_t \int_t^T M_u C_u \, \mathrm{d}u \\ &= \frac{1}{M_t} \mathsf{E}_t \int_t^T M_u (\gamma X_u - \xi)^+ \, \mathrm{d}u \\ &= \mathsf{E}_t \int_t^T \mathrm{e}^{-\delta(u-t)} \left(\frac{X_u}{X_t}\right)^{-\rho} (\gamma X_u - \xi)^+ \, \mathrm{d}u . \end{aligned} \tag{12}$$

Given the optimal wealth defined in (12), market completeness guarantees that there exists a portfolio process π satisfying the intertemporal budget constraint:

$$\mathrm{d}W_t = -C_t \, \mathrm{d}t + W_t [r \, \mathrm{d}t + \pi_t (\mu - r) \, \mathrm{d}t + \pi_t \sigma \, \mathrm{d}B_t] . \tag{13}$$

It is convenient to introduce some additional notation, in order to provide explicit formulas for the optimal W and π. Define

$$f(t, x) = \mathsf{E}\left[\int_t^T \mathrm{e}^{-\delta(u-t)} \left(\frac{X_u}{x}\right)^{-\rho} (\gamma X_u - \xi)^+ \, \mathrm{d}u \;\middle|\; X_t = x \right] , \tag{14}$$

where X is the geometric Brownian motion defined in (10). Because X is a geometric Brownian motion, we can calculate the conditional expectation (14) explicitly. Let N denote the standard normal distribution function and, for all $t < u$ and z and any constant α, define

$$\begin{aligned} V(t, z, \xi, \alpha) = \int_t^T & \mathrm{e}^{[-\delta + \alpha(r - \delta + \lambda^2/2)/\rho + \alpha^2 \lambda^2 / 2\rho^2](u-t)} \\ & \times \mathrm{N}\left(\frac{\rho(\log z - \log \xi) + (r - \delta + \lambda^2/2 + \alpha \lambda^2/\rho)(u-t)}{\lambda \sqrt{u-t}} \right) \mathrm{d}u . \end{aligned} \tag{15}$$

We take $\mathrm{N}(\infty) = 1$ to accommodate $\xi = 0$. Set $A(t, z, \xi) = V(t, z, \xi, 1 - \rho)$ and $B(t, z, \xi) = V(t, z, \xi, -\rho)$. The subscripts x and z in the following denote partial derivatives.

Proposition 1 *There exists a constant $\gamma > 0$ such that the the investor's optimal consumption at date t is $C_t = (\gamma X_t - \xi)^+$. Set $Z_t = \gamma X_t$. The optimal wealth at date t is*

$$W_t = f(t, X_t) = Z_t A(t, Z_t, \xi) - \xi B(t, Z_t, \xi) , \tag{16}$$

and the optimal portfolio at date t is

$$\pi_t = \frac{\mu - r}{\rho \sigma^2} \cdot \frac{X_t f_x(t, X_t)}{f(t, X_t)} \tag{17}$$

$$= \frac{\mu - r}{\rho \sigma^2} \cdot \frac{Z_t \big[A(t, Z_t, \xi) + Z_t A_z(t, Z_t, \xi) - \xi B_z(t, Z_t, \xi) \big]}{W_t} . \tag{18}$$

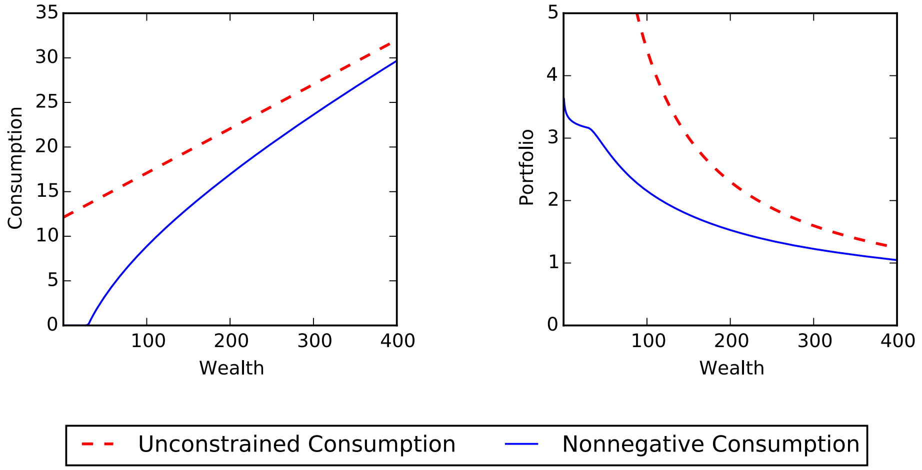

Fig. 2 Optimal consumption and portfolio. The solid lines depict the optimal consumption and portfolio described in Proposition 1 for an investor with a remaining horizon of 30 years when $r = 0.02$, $\mu = 0.08$, $\sigma = 0.20$, and $\delta = 0.05$. The investor's preference parameters are $\rho = 8$ and $\xi = 100$. The dashed lines depict the optimal consumption and portfolio for the same investor when negative consumption and wealth are allowed ("Appendix B")

If $\xi = 0$, then $xf_x(t, x) = f(t, x)$, and the optimal portfolio is $\pi = (\mu - r)/\rho\sigma^2$.

The dynamic programming approach to portfolio choice [13] provides the result that, when the investment opportunity set is constant and denoting the value function by J, the optimal portfolio is

$$\pi_t = \frac{\mu - r}{\sigma^2} \cdot \left(\frac{J_w(t, W_t)}{-W_t J_{ww}(t, W_t)} \right).$$

The expression in parentheses in this equation is the reciprocal of the relative risk averison of the value function. While we have not solved for the value function, Eq. (17) shows that the relative risk aversion of the value function is

$$\frac{-W_t J_{ww}(t, W_t)}{J_w(t, W_t)} = \frac{\rho f(t, X_t)}{X_t f_x(t, X_t)}.$$

The optimal consumption and optimal portfolio (18) are plotted as a function of wealth in a numerical example in Fig. 2. For the sake of comparison, we also plot optimal consumption and the optimal portfolio when negative consumption and wealth are allowed ("Appendix B"). When negative consumption and wealth are allowed, the investor consumes more when wealth is positive and takes on more risk. As the figure illustrates, the effect of the nonnegativity constraint is larger when wealth is lower. As can be seen from the figure, the kink in the optimal portfolio with the nonnegativity constraint occurs at the wealth level at which the constraint becomes binding.

3 Age and portfolio risk

We study three different investors having different preferences. We use a base case of $\rho = 4$ and $\xi = 0$. We also consider $\rho = 8$ and $\rho = 20$. We choose values of ξ for these values of ρ

so that the three investors all hold the same initial portfolio. We assume that the investors first solve the consumption/investment problem at age 20 and anticipate death at age 85. We use a real risk-free rate of $r = 2\%$ and a subjective discount rate of $\delta = 5\%$. We assume the risky asset has an expected rate of return of $\mu = 8\%$ and a standard deviation of $\sigma = 20\%$. We normalize $W_0 = 100$. The CRRA investor with $\rho = 4$ always invests $(\mu - r)/\rho\sigma^2 = 0.375$ in the risky asset. This allocation is matched at age 20 by the other investors when $\rho = 8$ and $\xi = 2.8$ and when $\rho = 20$ and $\xi = 11.5$.

Our main mode of analysis in this section is simulation, but first we illustrate possible portfolio trajectories by considering three specific market return paths. Equations (4) and (5) and Itô's formula imply that over any discrete time period of length Δt, we have

$$\Delta \log M = \left(-r - \frac{1}{2}\lambda^2 + \frac{\lambda\mu}{\sigma} - \frac{1}{2}\lambda\sigma\right) \Delta t - \frac{\lambda}{\sigma}\, \Delta \log S. \tag{19}$$

For a time period of one year, $\mathsf{E}[\Delta \log S] = \mu - \sigma^2/2$. Figure 3 shows the optimal wealth and portfolio of the three investors in three different examples. The top panel shows how wealth and portfolios evolve when market returns are higher than anticipated (the change in $\log S$ each year is $2\mathsf{E}[\Delta \log S]$). The middle panel is an example in which market returns are average each year (the change in $\log S$ each year is $\mathsf{E}[\Delta \log S]$). The bottom panel is an example in which market returns are below average each year (the change in $\log S$ each year is zero). We compute the optimal portfolio and optimal wealth each year (in the figure, we interpolate between years, producing smooth paths; in reality, the paths have nonzero quadratic variation).

We plot wealth in Fig. 3 in terms of the annuity that it will finance over the investor's remaining lifetime; that is, we plot $rW_t/(1 - \mathrm{e}^{-r(T-t)})$. This seems more informative than plotting wealth, which always decreases to zero over time, due to the finite horizon and absence of a bequest motive. In the top two panels, the favorable market returns cause the annuity value of the investor's wealth to increase over time. The lock-in effect therefore causes the IRRA investors to switch to more conservative portfolios over time. However, the reverse occurs in the bottom panel. The poor market returns in that example lead to declining annuity values, which cause IRRA investors to choose more aggressive portfolios.

Proposition 1 and Fig. 3 demonstrate that the optimal portfolio π_t of an IRRA investor at any date t is a random variable, depending on the market returns prior to t. We use Monte Carlo analysis to calculate the distribution of π_t. We draw 10,000 realizations of π_t by simulating 10,000 market lifetimes. Figure 4 plots the estimated mean of π_t as well as the estimated mean wealth and mean consumption at each age $t = 20, 21, \ldots, 84$ for each of the three investors (we omit age 85 because the optimal portfolio is undefined at $t = T$, due to the fact that $W_T = 0$).

The left panel of Fig. 4 shows that $\mathsf{E}[\pi_t]$ declines with t for IRRA investors. This is our main result. The middle panel of Fig. 4 helps to explain the result. We again plot wealth in terms of its annuity value in that panel. On average, the annuity value of wealth rises with age, so investors can be fairly confident of higher future consumption as they age. This produces the lock-in effect discussed previously, in which IRRA investors switch to more conservative portfolios as they become confident of higher future consumption. The fact that consumption typically does rise with age is shown in the right panel of Fig. 4.

Figure 4 shows that IRRA investors in our simulation hold more conservative portfolios than does the CRRA investor and achieve less wealth and consumption. However, these facts are sensitive to the calibration and hence are less robust than is the fact that IRRA investors reduce risk as they age.

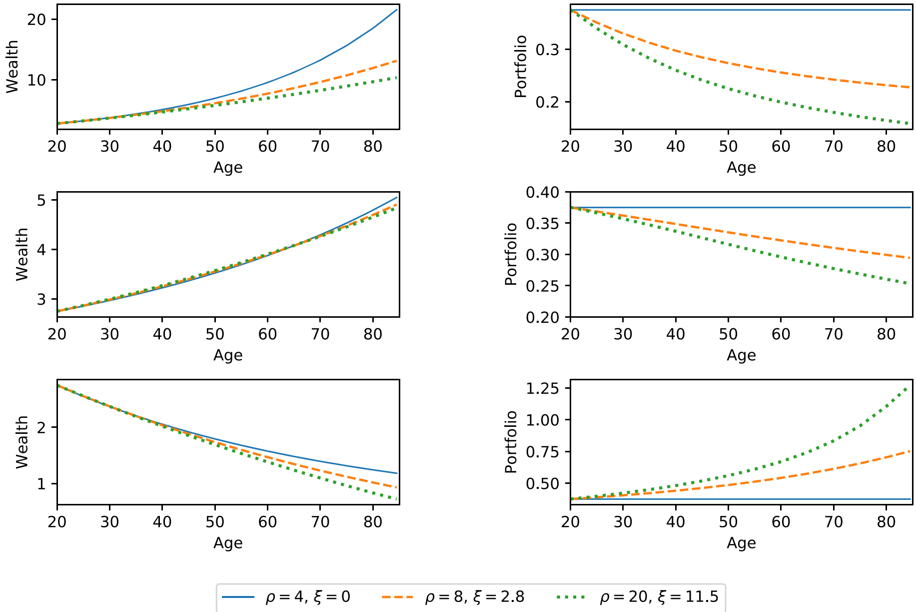

Fig. 3 Three possible paths. Using the parameterization of this section, three possible paths are generated. In the top panel, the continuously compounded market return $\Delta \log S$ is $2\mu - \sigma^2$ each year, in the middle panel it is $\mu - \sigma^2/2$ each year, and in the bottom panel it is 0 each year

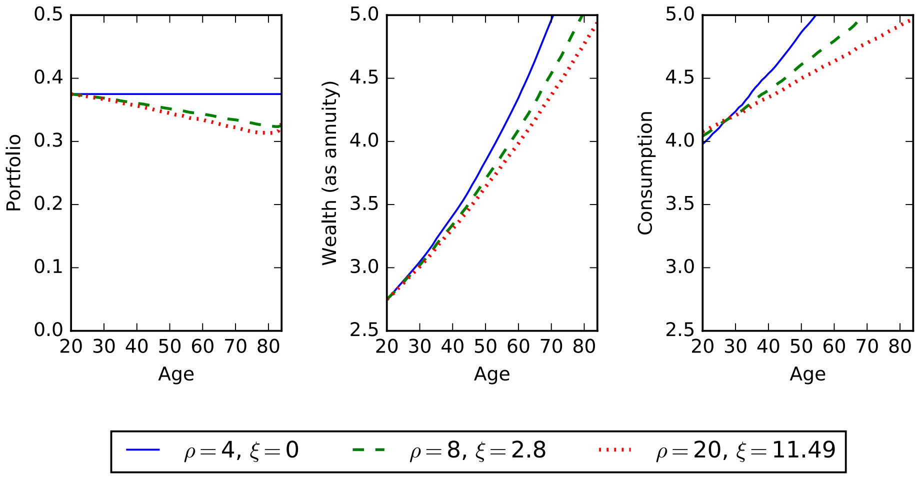

Fig. 4 Expected values computed by Monte Carlo. The market is simulated 10,000 times, and the consumption-investment problem is solved in each market for each of three (ρ, ξ) pairs. The figure displays the mean allocation to the risky asset, the mean wealth—measured as $rW/[1 - e^{-r(T-t)}]$—and mean consumption

As another illustration of how portfolio risk usually declines with age for IRRA investors, we regress the optimal portfolio on the investor's age, using annual observations. The regression coefficient is a random variable, depending on the market returns. We again use Monte Carlo analysis to calculate the distribution of the random variable. We use the same 10,000 simulated market lifetimes as before. Table 1 reports the results. Clearly, the optimal portfolio is typically decreasing with age for the IRRA investors. The mean coefficient of − 0.105 for $\rho = 20$ and $\xi = 11.5$ means that the optimal portfolio decreases by a bit more than a tenth of a percent per year on average, from 37.5% at age 20 to 30.8% at age 84.

4 Cross-sectional correlation of wealth and portfolio risk

One might think that the lock-in effect would produce a negative correlation between wealth and portfolio risk in the cross-section of investors, contrary to the positive correlation found in the data by Curcuru et al. [7] and others. However, this is not necessarily the case. If there is cross-sectional variation in risk preferences, then less risk-averse investors will invest more aggressively and accumulate higher wealth on average, producing a positive correlation between wealth and portfolio risk, despite the lock-in effect. To demonstrate this phenomenon, we solve the consumption-investment problem for 100 distinct investors, defined by a 10×10 grid on the (ρ, ξ) space. We arbitrarily use [2, 10] as the range for each parameter. At any date t, the cross-sectional correlation between wealth and portfolio risk is a random variable, depending on the market returns prior to t. We estimate the expected cross-sectional correlation by Monte Carlo analysis, using 1000 simulated market lifetimes. Table 2 reports the estimates of the expected cross-sectional correlation coefficients. The estimated mean is positive at each age. Thus, in the cross-section of investors, wealthier investors hold more aggressive portfolios on average, even when investors have IRRA preferences.

We should emphasize that we are not claiming that the positive cross-sectional correlation between wealth and portfolio risk is due to IRRA. In fact, results similar to those in Table 2 would hold for CRRA investors with heterogeneous risk preferences. Our point is that the lock-in effect, which implies a negative correlation between wealth and portfolio risk in the

Table 1 Monte Carlo analysis of the age-portfolio correlation

	Mean	t	SD	Min	25%	50%	75%	Max
$\rho = 8, \xi = 2.8$	− 0.085	− 33.97	0.25	− 0.52	− 0.18	− 0.13	− 0.05	12.02
$\rho = 20, \xi = 11.5$	− 0.105	− 19.00	0.55	− 0.86	− 0.27	− 0.19	− 0.08	18.90

The market is simulated 10,000 times. The optimal portfolio is regressed on age in each market realization for investors ($\rho = 8, \xi = 2.8$) and ($\rho = 20, \xi = 11.5$). The table records the distribution of regression coefficients across the 10,000 market paths and the t-statistics for the null hypothesis that the regression coefficient is zero

Table 2 Monte Carlo analysis of the wealth-portfolio correlation

	Age											
	5	10	15	20	25	30	35	40	45	50	55	60
Mean	0.16	0.16	0.15	0.26	0.24	0.27	0.29	0.29	0.36	0.38	0.41	0.45
SE	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03

The market is simulated 1000 times. Each market consists of 100 individuals defined by all combinations of 10 values of ρ uniformly distributed between 2 and 10 and 10 values of ξ uniformly distributed between 2 and 10. The cross-sectional correlation between wealth and the allocation to the risky asset is calculated at each age in each simulated market. The table reports the mean correlations across simulations and the standard errors of the means. The other parameters are $W_0 = 100$, $r = 0.02$, $\mu = 0.08$, $\sigma = 0.20$, $\delta = 0.05$, and $T = 65$

time series for individual investors, is perfectly consistent with a positive correlation between wealth and portfolio risk in the cross-section of investors.

5 Labor income and the optimal financial portfolio

The advice of financial planners discussed in the introduction is about investors' financial portfolios, whereas our results so far are on the allocation of total wealth to risky assets. In this section, in our complete markets setting, we derive a formula for an investor's optimal financial portfolio in terms of the optimal allocation of total wealth to risky assets that the previous part of the paper studies, in terms of the delta of the claim paying labor income, and in terms of the value of human capital relative to total wealth. This formula shows that, if the optimal allocation of total wealth to risky assets decreases over time due to increasing relative risk aversion, then the optimal allocation of financial wealth to stocks is also more likely to decrease over time, as advised by financial planners.

Let W^f denote financial wealth, and let L denote labor income. For each t, set

$$W_t^\ell = \frac{1}{M_t} \mathsf{E}_t \int_t^T M_u L_u \, \mathrm{d}u.$$

This is the present value of future labor income as of date t. Total wealth is $W = W^f + W^\ell$.

Let N denote the following martingale:

$$N_t = \mathsf{E}_t \int_0^T M_u L_u \, \mathrm{d}u.$$

By the martingale representation theorem, $\mathrm{d}N_t = \kappa_t \, \mathrm{d}B_t$ for some stochastic process κ. Define

$$\pi_t^\ell = \frac{\kappa_t W_t^\ell}{\sigma M_t} + \frac{\mu - r}{\sigma^2}.$$

A straightforward application of Itô's formula shows that

$$\mathrm{d}W_t^\ell = -L_t \, \mathrm{d}t + W_t^\ell \left[r \, \mathrm{d}t + \pi_t^\ell (\mu - r) \, \mathrm{d}t + \pi_t^\ell \sigma \, \mathrm{d}B_t \right]. \tag{20}$$

Thus, π_t^ℓ is the portfolio process that replicates the claim paying labor income. In other words, $\pi^\ell W^\ell / S$ is the delta of the claim paying labor income relative to the asset with price S. Holding $\pi^\ell W^\ell / S$ shares replicates the claim. The investor can short sell that many shares to hedge the labor income process.

Given the optimal total wealth process W and optimal portfolio π described in Sect. 2, the optimal financial wealth is $W_t^f = W_t - W_t^\ell$ and the optimal financial portfolio is π_f given by

$$\pi_t^f W_t^f + \pi_t^\ell W_t^\ell = \pi_t W_t. \tag{21}$$

The definitions of W^f and π^f in conjunction with (20) and (13) imply the intertemporal budget constraint

$$\mathrm{d}W_t^f = (L_t - C_t) \, \mathrm{d}t + W_t^f \left[r \, \mathrm{d}t + \pi_t^f (\mu - r) \, \mathrm{d}t + \pi_t^f \sigma \, \mathrm{d}B_t \right]. \tag{22}$$

The advice of financial planners regards π_f, whereas we have given results for π. Set $\phi_t = W_t^\ell / W_t$. From (21), we have

$$\pi_t^f = \frac{\pi_t - \phi_t \pi_t^\ell}{1 - \phi_t}. \tag{23}$$

Thus, results for π_t have implications for π_t^f. The precise implications depend on the labor income process and on the optimal wealth process W, which determines ϕ_t. The simplest special case is riskless labor income, in which case $\pi^\ell = 0$ and $\pi_t^f = \pi_t/(1-\phi_t)$. It is natural to assume ϕ decreases over time. In this case, π^f decreases over time even if π is constant [9,16]. However, π^f decreases faster if π also decreases, as we have shown is true on average for IRRA investors. Furthermore, if labor income is risky, then π^f may be constant or even increase over time when π is constant over time.[4] In general, we can see from (23) that, holding the paths of π^ℓ and ϕ fixed, there is a greater tendency for π^f to decrease over time if it is true that π decreases over time.

6 Conclusion

We show that IRRA utility functions are attractive, because they generate a more reasonable aversion to large gambles given a fixed aversion to small gambles. We also show that IRRA utility functions are tractable, even when negative consumption is disallowed. We solve the portfolio choice problem for IRRA investors in a continuous-time economy with a constant investment opportunity set and complete markets in semi-closed form. We show that the lock-in effect implies that agents with IRRA utility functions optimally decrease their total portfolio risk as they age, consistent with the standard advice of financial advisors. Furthermore, the cross-sectional correlation between wealth and equity allocations can be positive despite the lock-in effect. Our results are for total wealth and total portfolio risk, including the loading of human capital on the stock return, but our results also have implications for financial wealth and financial portfolio risk.

Appendix A. Proof of Proposition 1

The function

$$\gamma \mapsto \mathsf{E}\int_0^T M_t(\gamma X_t - \xi)^+ \,\mathrm{d}t$$

is strictly monotone and maps $[0,\infty)$ onto $[0,\infty)$. Thus, there is a unique γ such that

$$\mathsf{E}\int_0^T M_t(\gamma X_t - \xi)^+ \,\mathrm{d}t = W_0. \tag{A.1}$$

Let C^* denote $(\gamma X_t - \xi)^+$, and let C be any other nonnegative consumption process satisfying the budget constraint

$$\mathsf{E}\int_0^T M_t C_t \mathrm{d}t \le W_0. \tag{A.2}$$

[4] Huggett and Kaplan [8] document that there is substantial risk in labor income, though they estimate the value of the part that is spanned by asset markets to be less than 35% of the total value of human capital.

By concavity,

$$u(C) \le u(C^*) + u'(C^*)(C - C^*),$$

so

$$\mathsf{E}\int_0^T e^{-\delta t} u(C_t)\,dt \le \mathsf{E}\int_0^T e^{-\delta t} u(C_t^*)\,dt + \mathsf{E}\int_0^T e^{-\delta t} u'(C_t^*)(C_t - C_t^*)\,dt. \tag{A.3}$$

The second term on the right-hand side is the sum of

$$\mathsf{E}\int_0^T e^{-\delta t} u'(C_t^*)(C_t - C_t^*)1_{\{C_t^*>0\}}\,dt \tag{A.4}$$

and

$$\mathsf{E}\int_0^T e^{-\delta t} u'(C_t^*)(C_t - C_t^*)1_{\{C_t^*=0\}}\,dt. \tag{A.5}$$

The term (A.4) equals

$$\mathsf{E}\int_0^T e^{-\delta t} (\gamma X_t)^{-\rho}(C_t - C_t^*)1_{\{C_t^*>0\}}\,dt = \gamma^{-\rho}\mathsf{E}\int_0^T M_t(C_t - C_t^*)1_{\{C_t^*>0\}}\,dt. \tag{A.6}$$

The term (A.5) equals

$$\begin{aligned}\mathsf{E}\int_0^T e^{-\delta t} \xi^{-\rho}(C_t - C_t^*)1_{\{C_t^*=0\}}\,dt &\le \mathsf{E}\int_0^T e^{-\delta t} (\gamma X_t)^{-\rho}(C_t - C_t^*)1_{\{C_t^*=0\}}\,dt \\ &= \gamma^{-\rho}\mathsf{E}\int_0^T M_t(C_t - C_t^*)1_{\{C_t^*=0\}}\,dt, \end{aligned} \tag{A.7}$$

the inequality being due to the fact that $\xi \ge \gamma X_t$ when $C_t^* = 0$. Adding (A.6) and (A.7) and using (A.1) and (A.2) yields

$$\mathsf{E}\int_0^T e^{-\delta t} u'(C_t^*)(C_t - C_t^*)\,dt \le 0.$$

Hence, (A.3) implies that C^* is optimal.

Because C^* is the optimal consumption process, the expression (12)–which equals $f(t, X_t)$—is the optimal wealth process. We have

$$\begin{aligned} f(t,x) &= \int_t^T e^{-\delta(u-t)}\mathsf{E}\left[\left(\frac{X_u}{x}\right)^{-\rho}(\gamma X_u - \xi)^+ \mid X_t = x\right] du \\ &= \gamma x \int_t^T e^{-\delta(u-t)}\mathsf{E}\left[\left(\frac{X_u}{x}\right)^{1-\rho} 1_{\{X_u>\xi/\gamma\}} \mid X_t = x\right] du \\ &\quad -\xi \int_t^T e^{-\delta(u-t)}\mathsf{E}\left[\left(\frac{X_u}{x}\right)^{-\rho} 1_{\{X_u>\xi/\gamma\}} \mid X_t = x\right] du \end{aligned} \tag{A.8}$$

Consider any constant α and dates $t < u$. Set $\tau = u - t$. Using the definition of X and the fact that M is the geometric Brownian motion (5), we have

$$\log X_u - \log X_t = \frac{(r-\delta+\lambda^2/2)\tau}{\rho} + \frac{\lambda}{\rho}(B_u - B_t). \tag{A.9}$$

Therefore,

$$\left(\frac{X_u}{X_t}\right)^\alpha = \mathrm{e}^{\alpha(r-\delta+\lambda^2/2)\tau/\rho}\mathrm{e}^{-\alpha\lambda\sqrt{\tau}\varepsilon/\rho},$$

where $\varepsilon = -(B_u - B_t)/\sqrt{u-t}$, which is a standard normal random variable. Furthermore,

$$\begin{aligned} X_u \geq \frac{\xi}{\gamma} \quad &\Leftrightarrow \quad \log X_t + \frac{(r-\delta+\lambda^2/2)\tau}{\rho} - \frac{\lambda\sqrt{\tau}\varepsilon}{\rho} \geq \log\left(\frac{\xi}{\gamma}\right) \\ . \quad &\Leftrightarrow \quad \frac{\lambda\sqrt{\tau}\varepsilon}{\rho} \leq \log(\gamma X_t) - \log\xi + \frac{(r-\delta+\lambda^2/2)\tau}{\rho} \\ &\Leftrightarrow \quad \varepsilon \leq \omega, \end{aligned}$$

where we set

$$\omega = \frac{\rho[\log(\gamma X_t) - \log\xi]}{\lambda\sqrt{\tau}} + \frac{(r-\delta+\lambda^2/2)\sqrt{\tau}}{\lambda}.$$

Therefore,

$$\begin{aligned} \mathsf{E}\left[\left(\frac{X_u}{x}\right)^\alpha 1_{\{X_u>\xi/\gamma\}} \mid X_t = x\right] &= \mathrm{e}^{\alpha(r-\delta+\lambda^2/2)\tau/\rho}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega}\mathrm{e}^{-y^2/2-\alpha\lambda\sqrt{\tau}y/\rho}\,\mathrm{d}y \\ &= \mathrm{e}^{\alpha(r-\delta+\lambda^2/2)\tau/\rho+\alpha^2\lambda^2\tau/2\rho^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega}\mathrm{e}^{-(y+\alpha\lambda\sqrt{\tau}/\rho)^2/2}\,\mathrm{d}y \\ &= \mathrm{e}^{\alpha(r-\delta+\lambda^2/2)\tau/\rho+\alpha^2\lambda^2\tau/2\rho^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega+\alpha\lambda\sqrt{\tau}/\rho}\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\ &= \mathrm{e}^{\alpha(r-\delta+\lambda^2/2)\tau/\rho+\alpha^2\lambda^2\tau/2\rho^2}\,\mathrm{N}\left(\omega + \frac{\alpha\lambda\sqrt{\tau}}{\rho}\right). \end{aligned} \tag{A.10}$$

Substituting (A.10) with $\alpha = 1-\rho$ and $\alpha = -\rho$ into (A.8) verifies (16).

From the formula (16), it is straightforward to verify that f is continuously differentiable in t and twice continuously differentiable in x. Therefore, we can apply Itô's formula to compute $\mathrm{d}f$. We compute the optimal portfolio π by matching the stochastic part of $\mathrm{d}f$ to $W\pi\sigma\,\mathrm{d}B$, which is the stochastic part of $\mathrm{d}W$ implied by the intertemporal budget constraint. The drift parts will then match due to the fact that

$$\int_0^t M_s C_s\,\mathrm{d}s + M_t f(t, X_t)$$

is a martingale (which implies that its drift is zero). From (A.9), X is a geometric Brownian motion with volatility λ/ρ. Therefore, the stochastic part of $\mathrm{d}f$ is $(\lambda/\rho)Xf_x\,\mathrm{d}B$. Matching this to $W\pi\sigma\,\mathrm{d}B$, we see that

$$\pi_t = \frac{\lambda}{\rho\sigma}\cdot\frac{X_t f_x(t, X_t)}{W_t} = \frac{\mu - r}{\rho\sigma^2}\cdot\frac{X_t f_x(t, X_t)}{f(t, X_t)}.$$

This verifies (17). The formula (16) for f and (17) directly imply (18).

If $\xi = 0$, then the definition (14) simplifies to

$$f(t,x) = \gamma x \mathsf{E}\left[\int_t^T \mathrm{e}^{-\delta(u-t)}\left(\frac{X_u}{x}\right)^{1-\rho} \mathrm{d}u \;\middle|\; X_t = x\right],$$

which equals $xa(t)$, where $a(\cdot)$ is the nonrandom function

$$a(t) = \mathsf{E}\left[\int_t^T \mathrm{e}^{-\delta(u-t)}\left(\frac{X_u}{x}\right)^{1-\rho} \mathrm{d}u \;\middle|\; X_t = x\right].$$

Therefore, $f(t,x) = xf_x(t,x)$ as claimed.

Appendix B. Negative consumption

Here we derive the optimum for an IRRA investor when negative consumption is allowed. The first order condition is

$$\mathrm{e}^{-\delta t}(\xi + C_t)^{-\rho} = \eta M_t$$

As before, define $\gamma = \eta^{-1/\rho}$ and define X as in (10). Then, the first order condition can be expressed as: $C_t = \gamma X_t - \xi$. Optimal wealth is

$$\begin{aligned} W_t &= \mathsf{E}_t \int_t^T \frac{M_u}{M_t} C_u \,\mathrm{d}u \\ &= \gamma X_t \mathsf{E}_t \int_t^T \frac{M_u}{M_t}\frac{X_u}{X_t}\,\mathrm{d}u - \xi \mathsf{E}_t \int_t^T \frac{M_u}{M_t}\,\mathrm{d}u \\ &= \gamma X_t \mathsf{E}_t \int_t^T \mathrm{e}^{-\delta(u-t)/\rho}\left(\frac{M_u}{M_t}\right)^{(\rho-1)/\rho} \mathrm{d}u - \frac{\xi}{r}\left[1 - \mathrm{e}^{-r(T-t)}\right] \end{aligned}$$

Itô's formula and the dynamics of M imply

$$\begin{aligned} \frac{\mathrm{d}M^{(\rho-1)/\rho}}{M^{(\rho-1)/\rho}} &= \frac{\rho-1}{\rho}\frac{\mathrm{d}M}{M} + \frac{1-\rho}{2\rho^2}\left(\frac{\mathrm{d}M}{M}\right)^2 \\ &= \frac{1-\rho}{\rho} r\,\mathrm{d}t + \frac{1-\rho}{\rho}\lambda\,\mathrm{d}B + \frac{1-\rho}{2\rho^2}\lambda^2\,\mathrm{d}t \end{aligned}$$

Therefore,

$$\mathsf{E}_t\left[\left(\frac{M_u}{M_t}\right)^{(\rho-1)/\rho}\right] = \mathrm{e}^{[(1-\rho)r/\rho + (1-\rho)\lambda^2/2\rho^2](u-t)}.$$

It follows that

$$W_t = \frac{\gamma X_t}{\phi}\left[1 - \mathrm{e}^{-\phi(T-t)}\right] - \frac{\xi}{r}\left[1 - \mathrm{e}^{-r(T-t)}\right] \tag{B.1}$$

where

$$\phi = r - \frac{r-\delta}{\rho} - \frac{(1-\rho)\lambda^2}{2\rho^2}.$$

Define

$$\widehat{W}_t = W_t + \frac{\xi}{r}\left[1 - \mathrm{e}^{-r(T-t)}\right].$$

This is the investor's wealth plus the proceeds that would be obtained by selling a claim that pays ξ per unit of time from t to T. Equation (B.1) allows us to calculate γ from $\widehat{W}_0$ as

$$\gamma = \frac{\phi \widehat{W}_0}{1 - \mathrm{e}^{-\phi T}} \tag{B.2}$$

The optimal consumption satisfies

$$C_t + \xi = \gamma X_t = \frac{\phi \widehat{W}_t}{1 - \mathrm{e}^{-\phi (T-t)}}$$

The optimal portfolio π is such that $W \pi \sigma \, \mathrm{d}B$ equals the stochastic part of $\mathrm{d}W$, which is the stochastic part of $\mathrm{d}\widehat{W}$, which is the stochastic part of

$$\frac{\gamma}{\phi} \left[1 - \mathrm{e}^{-\phi (T-t)} \right] \mathrm{d}X$$

Thus,

$$\begin{aligned} W \pi \sigma &= \frac{\gamma \lambda}{\phi \rho} \left[1 - \mathrm{e}^{-\phi (T-t)} \right] X_t \\ &= \frac{\lambda}{\rho} \widehat{W}_t \end{aligned}$$

This implies

$$\pi = \frac{\widehat{W}_t}{W_t} \cdot \frac{\mu - r}{\rho \sigma^2}$$

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