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A characterization of the coskewness–cokurtosis pricing model

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HIGHLIGHTS

- If there are no "extremely attractive returns", the coskewness–cokurtosis pricing model holds.
- The converse also holds.
- An extremely attractive return has a positive alpha.
- An extremely attractive return also has residual risk with desirable coskewness and cokurtosis.

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ABSTRACT

The coskewness–cokurtosis pricing model is equivalent to absence of any positive-alpha return for which the residual risk has positive coskewness and negative cokurtosis with the market. This parallels the CAPM and also the fundamental theorem of asset pricing.

1. Introduction

The coskewness–cokurtosis pricing model of Kraus and Litzenberger (1976) and Dittmar (2002) extends the capital asset pricing model (CAPM) of Sharpe (1964), Treynor (1999), Lintner (1969), and Mossin (1966) by including coskewness and cokurtosis in addition to covariance with the market return as priced risks. The motivation for the model is that investors may care about skewness and kurtosis in addition to mean and variance. If so, then investors who hold the market portfolio would evaluate a marginal change in the holding of an asset in terms of its effect on variance, skewness, and kurtosis, and these marginal effects are captured by covariance, coskewness, and cokurtosis. The model asserts that each return R and the corresponding expected return $\bar{R}$ satisfy

$$\bar{R} - R_f = \lambda_1 \text{cov}(R, R_m) - \lambda_2 \text{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) + \lambda_3 \text{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right), \quad (1)$$

where R_f is the risk-free return, R_m is the market return, $\bar{R}_m$ is the expected market return, and $\lambda_0, \lambda_1, \lambda_2 > 0$. The signs of the coefficients in (1) are based on the assumption that investors dislike variance, prefer positive skewness to negative skewness, and dislike kurtosis.[1] Thus, high covariance/low coskewness/high cokurtosis assets are undesirable and consequently sell at low prices, producing high expected returns.

Consider the projection of an excess return $R - R_f$ on the market excess return $R_m - R_f$:

$$R - R_f = \alpha + \beta\left(R_m - R_f\right) + \varepsilon, \quad (2)$$

where $\beta = \text{cov}(R, R_m)/\text{var}(R_m)$ and where ε has mean zero. It is easy to see that the coskewness–cokurtosis pricing model implies that there cannot be any returns possessing all of the following properties:

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[1] This is consistent with utility functions being concave with positive third derivatives and negative fourth derivatives, conditions that follow from risk aversion, decreasing absolute risk aversion, and decreasing absolute prudence (Arditti, 1967; Kimball, 1993; Haas, 2007).

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(i) $\alpha > 0$,
(ii) $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] > 0$, and
(iii) $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right] < 0$.

If such a return were to exist, it would be a positive alpha return with a residual risk having positive coskewness and negative cokurtosis with the market. Such a return would be extremely attractive to investors with mean–variance–skewness–kurtosis preferences who hold the market portfolio, and the coskewness–cokurtosis pricing model implies that such extremely attractive returns cannot exist. This is parallel to the CAPM, which implies that positive alpha returns cannot exist.

The contribution of this note is to show that the absence of returns with properties (i)–(iii) is a sufficient as well as necessary condition for the coskewness–cokurtosis pricing model. Thus, we derive the model from the absence of returns that are extremely attractive to investors with mean–variance–skewness–kurtosis preferences who hold the market portfolio. We do not need to assume that there is a representative investor whose utility function can be well approximated by a Taylor series expansion as do Kraus and Litzenberger (1976) and Dittmar (2002), though of course our motivation for assuming the absence of returns satisfying (i)–(iii) is not far from their representative investor hypothesis.

To put the result in perspective, it is useful to consider three separate classes of securities markets:

$\mathcal{M}_1$: markets in which there is no return satisfying (i),
$\mathcal{M}_2$: markets in which there is no return satisfying both (i) and (ii),
$\mathcal{M}_3$: markets in which there is no return satisfying all of (i)–(iii).

Clearly, $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$. The CAPM holds in the smallest set of markets $\mathcal{M}_1$. The coskewness pricing model holds in the middle set $\mathcal{M}_2$ (the coskewness model is the coskewness–cokurtosis model with a zero coefficient on cokurtosis). The coskewness–cokurtosis pricing model holds in the largest set $\mathcal{M}_3$.

As with all factor pricing models, the coskewness–cokurtosis pricing model is equivalent to an affine representation of a stochastic discount factor (SDF). An SDF is a random variable M such that $\mathsf{E}[MR] = 1$ for all (gross) returns R. The coskewness–cokurtosis pricing model holds if and only if there is an SDF of the form

$$M = a_0 - a_1 R_m + a_2 \left(R_m - \bar{R}_m\right)^2 - a_3 \left(R_m - \bar{R}_m\right)^3, \tag{3}$$

for constants a_i with $a_1, a_2, a_3 > 0$. Thus, the result of this note can be expressed as: there are no returns satisfying (i)–(iii) if and only if there is an SDF of the form (3). This is analogous to what Dybvig and Ross (1989) call the fundamental theorem of asset pricing. The fundamental theorem of asset pricing states that the following are equivalent: (1) absence of arbitrage opportunities, (2) existence of a strictly positive SDF, and (3) existence of an optimum for an investor with monotone utility. The absence of returns satisfying (i)–(iii) imposes more structure on the space of returns than does the absence of arbitrage opportunities. Consequently, the absence of returns satisfying (i)–(iii) has strong implications for the nature of the SDF.[2] The key step in the proof of the theorem below and the key step in the proof that (1) implies (2) in the fundamental theorem of asset pricing are the same, namely, they invoke a version of the separating hyperplane theorem.

[2] Note, however, that, whereas the absence of arbitrage opportunities implies the existence of a strictly positive SDF, the SDF (3) cannot be strictly positive unless the market return is bounded. See Dybvig and Ingersoll (1982) for a discussion of this point in connection with the CAPM.

2. Main result

Suppose there is a risk-free asset with return R_f and N risky assets with returns R_i. By "return", we mean gross return, that is, one plus the rate of return. The set of returns is the set of random variables

$$R = R_f + \sum_{i=1}^{N} w_i (R_i - R_f), \tag{4}$$

for $w = (w_1, \dots, w_N) \in \mathbb{R}^N$. Assume that each of the returns R_i has a finite fourth moment, so all of the returns (4) also have finite fourth moments. As previously remarked, the equivalence of (b) and (c) below is a standard result, but we include it for the sake of completeness.

Theorem. *The following statements are equivalent:*

(a) *There does not exist a return R satisfying*
(i) $\alpha \geq 0$,
(ii) $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] \geq 0$, *and*
(iii) $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right] \leq 0$,
with at least one strict inequality, where α and ε are defined from R by the projection (2).
(b) *There exist $\lambda_0, \lambda_1, \lambda_2 > 0$ such that the coskewness–cokurtosis pricing model (1) holds for all returns R.*
(c) *There exist a_0 and $a_1, a_2, a_3 > 0$ such that M defined by (3) is an SDF.*

3. Proof of the theorem

For any $w \in \mathbb{R}^N$, the return R defined by (4) has an alpha equal to

$$\alpha = \sum_{i=1}^{N} w_i \alpha_i,$$

where α_i is the alpha of the return R_i in the projection (2) for $i = 1, \dots, N$. Furthermore, the residual risk of R is

$$\varepsilon = \sum_{i=1}^{N} w_i \varepsilon_i,$$

where ε_i is the residual risk of the return R_i in the projection (2). Let $A_1 = (\alpha_1 \cdots \alpha_N)'$. Let A_2 denote the column vector formed by stacking the numbers $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon_i\right]$ for $i = 1, \dots, N$. Let A_3 denote the vector formed by stacking the numbers $\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon_i\right]$ for $i = 1, \dots, N$. Now, let A denote the $N \times 3$ matrix that has A_1, A_2, and $-A_3$ as its columns. Given $w \in \mathbb{R}^N$, we have

$$A'w = \begin{pmatrix} \alpha \\ \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] \\ -\mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right] \end{pmatrix},$$

where R is the return (4) with α and ε defined by the projection (2).

Assume (a) holds. Then, the set $\{A'w \mid w \in \mathbb{R}^N\}$ intersects the positive orthant of $\mathbb{R}^3$ only at the origin. By Tucker's Complementarity Theorem (Rockafellar, 1970, Theorem 22.7), it follows that there exist $\eta_1, \eta_2, \eta_3 > 0$ such that $\eta' A' w \leq 0$ for all $w \in \mathbb{R}^N$, where $\eta = (\eta_1\ \eta_2\ \eta_3)'$. Because the set $\{A'w \mid w \in \mathbb{R}^N\}$ is a linear subspace of $\mathbb{R}^3$, this implies $\eta' A' w = 0$ for all $w \in \mathbb{R}^N$. Thus, for any $w \in \mathbb{R}^N$ with corresponding return (4) and projection (2),

$$\eta_1 \alpha + \eta_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] - \eta_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right] = 0. \tag{5}$$

Define $\lambda_2 = \eta_2/\eta_1$ and $\lambda_3 = \eta_3/\eta_1$. Now, we have

$$\alpha = -\lambda_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] + \lambda_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right]. \tag{6}$$

This implies

$$\begin{aligned}\bar{R} - R_f &= \alpha + \beta\left(\bar{R}_m - R_f\right) \\ &= -\lambda_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2 \varepsilon\right] + \lambda_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3 \varepsilon\right] \\ &\quad + \beta\left(\bar{R}_m - R_f\right).\end{aligned}$$

Substituting

$$\varepsilon = R - \bar{R} - \beta\left(R_m - \bar{R}_m\right)$$

gives

$$\begin{aligned}\bar{R} - R_f &= -\lambda_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) + \lambda_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right) \\ &\quad + \beta\left\{\lambda_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3\right]\right. \\ &\quad \left. - \lambda_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^4\right] + \bar{R}_m - R_f\right\}.\end{aligned}$$

Define

$$\lambda_1 = \frac{\lambda_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3\right] - \lambda_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^4\right] + \bar{R}_m - R_f}{\mathrm{var}\left(R_m\right)}. \tag{7}$$

Then, we have

$$\begin{aligned}\bar{R} - R_f &= -\lambda_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right) + \beta\lambda_1 \mathrm{var}\left(R_m\right) \\ &= \lambda_1 \mathrm{cov}\left(R, R_m\right) - \lambda_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right).\end{aligned}$$

Thus, (a) implies (b).

Now, assume (b) holds. Using (b) with $R = R_m$ gives

$$\begin{aligned}\bar{R}_m - R_f &= \lambda_1 \mathrm{var}\left(R_m\right) - \lambda_2 \mathrm{cov}\left(R_m, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(R_m, \left(R_m - \bar{R}_m\right)^3\right).\end{aligned} \tag{8}$$

Using (b) for an arbitrary return R gives

$$\begin{aligned}\alpha &= \bar{R} - R_f - \beta\left(\bar{R}_m - R_f\right) \\ &= \lambda_1 \mathrm{cov}\left(R, R_m\right) - \lambda_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right) - \beta\left(\bar{R}_m - R_f\right).\end{aligned}$$

Now substitute

$$R = \bar{R} + \beta\left(R_m - \bar{R}_m\right) + \varepsilon$$

and use $\mathrm{cov}\left(\varepsilon, R_m\right) = 0$ to obtain

$$\begin{aligned}\alpha &= \beta\left[\lambda_1 \mathrm{var}\left(R_m\right) - \lambda_2 \mathrm{cov}\left(R_m, \left(R_m - \bar{R}_m\right)^2\right)\right. \\ &\quad \left. + \lambda_3 \mathrm{cov}\left(R_m, \left(R_m - \bar{R}_m\right)^3\right)\right] \\ &\quad - \lambda_2 \mathrm{cov}\left(\varepsilon, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(\varepsilon, \left(R_m - \bar{R}_m\right)^3\right) - \beta\left(\bar{R}_m - R_f\right).\end{aligned}$$

Substituting (8), we obtain

$$\alpha = -\lambda_2 \mathrm{cov}\left(\varepsilon, \left(R_m - \bar{R}_m\right)^2\right) + \lambda_3 \mathrm{cov}\left(\varepsilon, \left(R_m - \bar{R}_m\right)^3\right). \tag{9}$$

Suppose the return satisfies (ii) and (iii); that is, its residual risk ε has nonnegative coskewness and nonpositive cokurtosis. Then, (9) implies that $\alpha \leq 0$. Thus, there are no returns satisfying (ii) and (iii) for which (i) holds with strict inequality. Furthermore, if the return satisfies (ii) and (iii) and there is strict inequality in either (ii) or (iii), then (9) implies $\alpha < 0$. Thus, there are no returns satisfying (i)–(iii) with strict inequality in one of the three conditions.

It remains to establish the equivalence of (b) and (c). First, suppose that (c) holds. The fact that $\mathsf{E}\left[MR_f\right] = 1$ implies $\bar{M} = 1/R_f$. For an arbitrary return R, the fact that $\mathsf{E}[MR] = 1$ implies

$$1 = \mathrm{cov}\left(M, R\right) + \bar{M}\bar{R} = \mathrm{cov}\left(M, R\right) + \frac{\bar{R}}{R_f}.$$

Thus,

$$\bar{R} = R_f - R_f \mathrm{cov}\left(M, R\right).$$

Now, substituting the form of M from (3), we have

$$\begin{aligned}\bar{R} &= R_f + R_f\left[a_1 \mathrm{cov}\left(R, R_m\right) - a_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right)\right. \\ &\quad \left. + a_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right)\right].\end{aligned}$$

Defining $\lambda_i = R_f a_i$ for $i = 1, 2, 3$ yields (b).

Now, suppose that (b) holds. Define

$$\begin{aligned}\xi &= \lambda_1\left(R_m - \bar{R}_m\right) - \lambda_2\left\{\left(R_m - \bar{R}_m\right)^2 - \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2\right]\right\} \\ &\quad + \lambda_3\left\{\left(R_m - \bar{R}_m\right)^3 - \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3\right]\right\}.\end{aligned}$$

Then, for any return R,

$$\begin{aligned}\mathsf{E}\left[\xi R\right] &= \lambda_1 \mathrm{cov}\left(R, R_m\right) - \lambda_2 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^2\right) \\ &\quad + \lambda_3 \mathrm{cov}\left(R, \left(R_m - \bar{R}_m\right)^3\right) \\ &= \bar{R} - R_f.\end{aligned}$$

Setting $M = 1/R_f - \xi/R_f$, we have

$$\mathsf{E}\left[MR\right] = \frac{\bar{R}}{R_f} - \frac{\mathsf{E}\left[\xi R\right]}{R_f} = 1.$$

Moreover, M is of the form (3) with

$$\begin{aligned}a_0 &= \frac{1}{R_f}\left\{1 + \lambda_1 \bar{R}_m - \lambda_2 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^2\right]\right. \\ &\quad \left. + \lambda_3 \mathsf{E}\left[\left(R_m - \bar{R}_m\right)^3\right]\right\},\end{aligned}$$

and $a_i = \lambda_i/R_f$ for $i = 1, 2, 3$. Thus, (c) holds.

4. Conclusion

This note establishes an equivalence between the absence of certain returns and the coskewness–cokurtosis pricing model. The equivalence may be useful for performance evaluation. Leland (1999) shows that a reliance on alphas for performance evaluation may induce managers to select negatively skewed returns. In the extreme, a return with negative skewness and excess kurtosis is called "picking up nickels before a steamroller". In order to ensure that a manager is not generating alpha by picking up nickels before a steamroller, it would be worthwhile to estimate the coskewness and cokurtosis of the manager's residual risk. If a manager can produce positive alpha while generating residual risk with positive coskewness and negative cokurtosis, then a strong case exists for investing in the manager. Some steps in this direction are made by Duarte et al. (2007), who estimate skewness as well as alphas of fixed income strategies. However, this seems to be the exception rather than the rule, and still does not address coskewness or cokurtosis of residual risks, so it does not directly address the existence of strategies having the properties (i)–(iii).

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